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A quantum fluids approach to frustrated Heisenberg models

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Abstract. Emphasising the close analogies between antiferromagnetism and neutral superfluidity, we develop a gauge-invariant quantum fluids description of non-bipartite Heisenberg systems. The antiferromagnet is treated as a spin superfluid with a rotational gauge invariance associated with the continuity of spin flow. We show how an extended Schwinger boson approach naturally incorporates the Onsager reaction fields generated by spin fluctuations, and correctly reproduces the semiclassical behaviour of spin wave theory in the large-S limit. The important short-wavelength physics of fluctuation-stabilised order is also captured by this description. For two-dimensional helimagnets at small S, the method predicts that the twist will survive the loss of sublattice magnetisation, closely analogous to the biaxial–uniaxial transition of nematic liquid crystals.

1. Introduction

The renewed interest in two-dimensional spin-$\frac{1}{2}$ Heisenberg models [1] demands a new description of strongly fluctuating quantum antiferromagnets [2–4], one which encompasses the semiclassical behaviour of spin wave theory *and* also survives the loss of sublattice magnetisation. Traditional spin wave methods are inapplicable when global spin rotation invariance is not broken, and to proceed we must develop a gauge-invariant approach to magnetism. Two advances have been made in this direction. Liang *et al* [5] have written the ground state wavefunction of a bipartite antiferromagnet in a singlet RVB form, where long-range antiferromagnetic order is generated by rather short-range spin pairing. In a related development, Arovas and Auerbach [6, 7], extending earlier work of Takahashi [8], have described two-dimensional bipartite magnets by incompressible fluids of paired spin $\frac{1}{2}$ bosons with density $2S$ per site. The essential observation here, due to Wigner and Schwinger [9], is that a spin S is faithfully represented by a symmetric wavefunction of $2S$ spin-$\frac{1}{2}$ bosons

$$\boldsymbol{S}_i = \tfrac{1}{2} b^{\dagger}{}_{i\sigma} \boldsymbol{\sigma}_{\sigma\sigma'} b_{i\sigma'} \qquad b^{\dagger}{}_{i\sigma} b_{i\sigma} = 2S \tag{1.1}$$

where $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices. Condensation of the Schwinger bosons reflects magnetic ordering [10], whereas spin fluctuations are described by the 'normal' fluid. This analogy between antiferromagnetism and superfluidity has been stressed

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previously for long-wavelength modes [11, 12, 13]; the Arovas–Auerbach results suggest a feasible quantum fluids approach to magnetism at *short* length scales. Here we exploit this analogy; the antiferromagnet is no longer treated as a rigid magnetic structure, but rather as a spin superfluid (figure 1.).

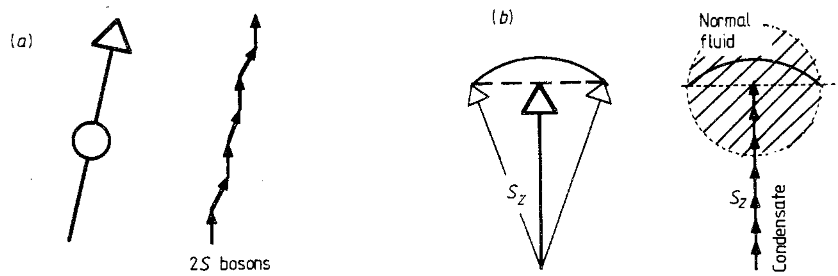

Figure 1. The semi-classical versus the quantum fluids approach to magnetism. (*a*) Illustration of how a spin S is built with $2S$ bosons. (*b*) Schematic diagram of the two fluid picture; the normal fluid describes the spin fluctuations while classical magnetism is the condensate.

Local continuity of flow, relating the current divergence and local density fluctuations, is an essential feature of fluidity. In a charged quantum fluid, local continuity is associated with local gauge invariance. Specifically, if Ψ is the complex order parameter, then under a gauge transformation

$$\begin{aligned} &\Psi(x) \rightarrow \exp(\mathrm{i}\theta(x))\Psi \\ &(\phi, \boldsymbol{A}) \rightarrow \left(\phi + \frac{\partial\theta}{\partial t}, \boldsymbol{A} + \nabla\theta\right) \end{aligned} \tag{1.2}$$

the condition that the free energy remain invariant establishes the local continuity

$$\frac{\delta F[\theta]}{\delta\theta(x)} = \frac{\partial\rho_{\mathrm{s}}(x)}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}(x) = 0. \tag{1.3}$$

In neutral superfluids, where currents do not couple to the physical electromagnetic vector potential, one can nevertheless introduce a fictitious vector potential solely as a book-keeping device to keep track of the current correlations [14]. In this case the free energy still displays a local gauge invariance, but under a gauge transformation the vector potential remains a pure gauge $\boldsymbol{A} = \boldsymbol{\nabla}\theta(x)$. A similar approach can be taken in quantum spin systems. Conventionally we attribute a Heisenberg antiferromagnet with *global* spin conservation associated with spin rotational invariance. However, as in the neutral superfluid, the motion of spin is a continuous process and there is a spin continuity equation relating divergence of spin currents to the precession of the local moments. In a completely analogous fashion to a U(1) superfluid, we can associate current conservation with a local gauge invariance of a curl-free spin vector potential. This fictitious field not only permits us to keep track of the spin currents, it also enables us to reparameterise our local spin coordinate axes in both space and time, provided that we also change the associated rotational gauge fields. For example, the long-wavelength action for a Heisenberg antiferromagnet in a magnetic field $\boldsymbol{B}$ is the non-linear sigma model [15,16],

$$I = \frac{S^2 J}{2} \int \mathrm{d}^d x \, \mathrm{d}t \left\{ \sum_i (\nabla_i \hat{\boldsymbol{n}} + \boldsymbol{A}_i \times \hat{\boldsymbol{n}})^2 - \frac{1}{c^2} \left(\frac{\partial \hat{\boldsymbol{n}}}{\partial t} + \boldsymbol{B} \times \hat{\boldsymbol{n}} \right)^2 \right\} \tag{1.4}$$

where $\boldsymbol{M} = S\hat{\boldsymbol{n}}$ is the staggered magnetisation, c is the spinwave velocity, $\boldsymbol{B}$ the magnetic field and $\boldsymbol{A}$ is a fictitious spin vector potential (pure gauge) used for determining the spin currents. The field dependence of the non-linear sigma model has been discussed by Andreev and Marchenko [15]. Equation (1.4) has the same form as the Landau–Ginzburg classical action for a conventional Bose superfluid

$$I = \frac{\rho_s}{2m} \int \mathrm{d}^d x \, \mathrm{d}t \left\{ \sum_i (\hbar \nabla_i \phi - qA_i)^2 - \frac{1}{c^2} \left(\hbar \frac{\partial \phi}{\partial t} - \mu \right)^2 \right\}. \tag{1.5}$$

Here ϕ is the phase of the condensate, ρ_s is the superfluid density, c is the speed of sound and the charge q is set to zero for a neutral system.

From a comparison of equations (1.4) and (1.5) we see that in a magnet the spin S plays the role as a superfluid density, $1/JS$ is the Bose mass, and the magnetic field acts as a chemical potential. In strict analogy with the U(1) superfluid, coherent spin currents result from the broken SU(2) rotational gauge invariance associated with global spin conservation [17]. These persistent spin currents

$$\boldsymbol{j}_i = -\delta I / \delta \boldsymbol{A}_i = JS^2 \not{\nabla}_i \hat{\boldsymbol{n}} \times \hat{\boldsymbol{n}} \qquad \not{\nabla}_i = (\nabla_i + \boldsymbol{A}_i \times) \tag{1.6}$$

correspond to twisted spin configurations relieved only by topological singularities or by spin flow from the boundaries. In equilibrium, the presence of a magnetic field $\boldsymbol{B}$ induces spin precession

$$\partial \hat{\boldsymbol{n}} / \partial t = -\boldsymbol{B} \times \hat{\boldsymbol{n}} \tag{1.7}$$

analogous to the precession of phase in a superfluid. Furthermore, a gradient in $\boldsymbol{B}$ produces a gradual increase in the spin twist, and thus a constant rate of change in the spin current

$$\not{\partial}_t \boldsymbol{j}_i = \mathrm{d}\boldsymbol{j}_i / \mathrm{d}t + \boldsymbol{B} \times \boldsymbol{j}_i = -JS^2 \boldsymbol{\nabla}_i \boldsymbol{B}_\perp \tag{1.8}$$

where $\boldsymbol{B}_\perp$ is the field component perpendicular to the staggered magnetisation. Equation (1.8) is the spin analogue of the Josephson equation

$$\mathrm{d}j_i / \mathrm{d}t = -(\hbar \rho_s / m) \nabla_i \mu. \tag{1.9}$$

We note, however, that a spin Meissner effect does not occur in Heisenberg systems; as in all neutral superfluids, the gauge fields have infinite stiffness. In table 1 we summarise the analogies between neutral and spin superfluidity.

A rotational invariant treatment of quantum antiferromagnetism must capture the essential physics of spin fluctuations, and in particular must extend the concept of the Weiss field to cases where the local moment vanishes. Such a generalisation has been discussed by Brout and Thomas [18, 19] in the context of disordered Ising magnets;

Table 1. A summary of the analogies between neutral and spin superfluidity.

	NEUTRAL SUPERFLUID	A.F.M./ SPIN SUPERFLUID
EXAMPLES	He-4 Spin-polarized H	La_2CuO_4, K_2NiF_4 Spin polarized H
ORDER PARAMETER	$\Psi = \sqrt{\rho_s}e^{i\phi}$	$M = \lvert M\rvert\hat{n}$
CONSERVED QUANTITY/ SYMMETRY	PARTICLE NO./ U(1)	SPIN/ SU(2)
PERSISTENT CURRENT	Particle current $j_i = \frac{\hbar\rho_s}{m}\nabla_i\phi$	Spin current $\vec{j}_i = JM^2\hat{n}\times\nabla_i\hat{n}$
CONJUGATE POTENTIAL	CHEMICAL POTENTIAL μ	MAGNETIC FIELD $\vec{B}$
STIFFNESS PARAMETER	LONDON KERNEL	SPIN STIFFNESS TENSOR
JOSEPHSON RELATION	$\partial_t j_i = -\frac{\hbar\rho_s}{m}\nabla_i\mu$	$(\partial_t + \vec{B}\times)\vec{j}_i = -JM^2\nabla_j\vec{B}_\perp$
COLLECTIVE MODES	First sound	Spin waves/ Twist wave
	Roton	Quantum exchange modes

they point out that Onsager's idea [20] of a non-orienting reaction field applies to spins. Specifically, a spin is sensitive to the field it would experience in an empty cavity. In the treatment of Brout and Thomas, this orienting field is the sum of the Weiss field in the uniform sample and the reaction field B_r of the empty cavity where

$$B_r(R_i) = -2\mu\langle S_i\rangle. \tag{1.10}$$

The Schwinger boson approach provides a microscopic realisation of B_r, which hitherto has not been stressed. Physically, the concept of a reaction field involves changing the total value of the spin and computing the resulting field

$$B_r = -J_{ij}\delta S_j \tag{1.11}$$

where J_{ij} is the exchange constant between sites i and j, and

$$\delta S_j = \langle S_j\rangle_{\text{cavity}} - \langle S_j\rangle_{\text{uniform}}. \tag{1.12}$$

is the change of magnetisation at site j due to the formation of a cavity at site i. Thus, the concept of a cavity reaction field exists within the context of a grand-canonical ensemble of spins.

Microscopically, an Onsager field is generated by a term in the Hamiltonian of the form

$$H_I = -\sum_i \mu_i(\tau)[S_i\cdot S_i - S(S+1)]. \tag{1.13}$$

Such an interaction is zero in the Gibbs ensemble of definite spin; however if we represent the same system by a grand-canonical ensemble of Schwinger bosons in a fluctuating chemical potential at fixed density, then the Onsager reaction field can be identified with the constraint. Formally we may rewrite the Onsager reaction field in the more familiar chemical potential form

$$\begin{aligned}\int_{\mu_o-i\infty}^{\mu_o+i\infty}\prod_{i,\tau}d\mu_i(\tau)\,\exp\left(-\int d\tau\,H_I(\tau)\right) &= \prod_{i,\tau}\delta[\boldsymbol{S}_i\cdot\boldsymbol{S}_i-S(S+1)]\\ &= \prod_{i,\tau}\delta[(n_i/2)(n_i/2+1)-S(S+1)]\\ &\doteq \prod_{i,\tau}\delta[(S+\tfrac{1}{2})(n_i-2S)]\\ &= \int_{\mu_o-i\infty}^{\mu_o+i\infty}\prod_{i\tau}d\mu_i(\tau)\,\exp\left(-\int_0^\beta d\tau\,H_I'(\tau)\right)\end{aligned} \tag{1.14}$$

where the constraint term is rewritten as

$$H_I' = -(S+\tfrac{1}{2})\sum_i \mu_i(\tau)[n_i-2S]. \tag{1.15}$$

In other words, a spin system of definite S can be modeled as a grand-canonical ensemble of spin bosons moving in the background of a fluctuating Onsager reaction field.

Following Brout and Thomas, in a linear response approximation [19] the Onsager reaction field is directly proportional to the magnetic fluctuation energy per site

$$\mu_j = [\tfrac{1}{3}S(S+1)]^{-1}\sum_j J_{ij}\langle\boldsymbol{S}_i\cdot\boldsymbol{S}_j\rangle_\mu. \tag{1.16}$$

A similar result appears in the mean field Schwinger boson approach. Since, in the absence of a sublattice magnetisation, $\langle\boldsymbol{S}_i\cdot\boldsymbol{S}_j\rangle$ is determined by fluctuations, a feedback exists that maintains the fluctuation dissipation theorem. By continuity, as an ordering transition is approached, the reaction field fluctuates more and more slowly; eventually freezing into the constant Weiss exchange field.

The Onsager reaction field was first applied to Heisenberg spin systems by Takahashi [8], who employed a reaction field to extend spin wave theory for the one dimensional ferromagnet to finite temperatures. Arovas and Auerbach subsequently employed an Onsager field in the context of a Schwinger boson approach to 1D ferromagnets and 2D antiferromagnets [6]. Takahashi later showed that his approach [21] could also be extended to antiferromagnets. Indeed, the results of the Schwinger boson and Takahashi methods are qualitatively similar in the paramagnetic phase: the choice of one method over the other is largely a question of taste and emphasis. Though we have chosen to employ a Schwinger boson approach, emphasising the rotationally invariant aspects of the problem, a similar line of development could be made with a generalisation of Takahashi's methods [8, 21].

A second aspect of strongly fluctuating spin systems concerns the physics of fluctuation-stabilised order. In non-bipartite magnets, spin wave fluctuations can select new forms of long-wavelength order from a manifold of degenerate classical configurations, Villain's 'order from disorder'. Here the short-wavelength spin fluctuations remove special phason modes present in classical magnets, producing small 'quantum exchange gaps' in the spinwave spectrum [22]. These new correlations develop independently of the ordered moment and are a sort of spin Van der Waals interaction. In the fluids analogy, these quantum exchange modes are the spin analogues of rotons in ^4He.

A testbed for the study of non-bipartite magnets is the two dimensional frustrated Heisenberg model

$$H = \frac{1}{2} \sum_{ij} J(\boldsymbol{R}_{ij}) \boldsymbol{S}_i \cdot \boldsymbol{S}_j \tag{1.17}$$

where the Fourier transform of $J(\boldsymbol{R})$ is

$$J(\boldsymbol{q}) = 2J_1(c_x + c_y) + 4J_2(c_x c_y) + 2J_3(c_{2x} + c_{2y}) + \cdots \tag{1.18}$$

in which $c_l = \cos q_l a$, $(l = x, y)$ and J_1, J_2, J_3 are first, second, and third nearest neighbour couplings respectively. Ioffe and Larkin [23] have emphasised that even if the higher order couplings J_q where $q \geq 3$ are originally zero, they will be generated by non-linear self-energy effects. Here we are interested in the case where the bare values of $J_1, J_2 \gg J_3$. In particular, in the region of the point $J_1 \sim 2J_2$ the system has a large classical degeneracy; here quantum fluctuations stabilise an incommensurate twisted phase with characteristic wavevector $\boldsymbol{Q}$ preferentially aligned along the x or y axes. At a large but finite critical spin value S_c the sublattice magnetisation is suppressed, leading to the absence of a local moment and a large-S liquid phase. A principal aim of this paper is to develop a method that will probe the spin correlations in all regions of this phase diagram, both with and without the presence of a sublattice magnetisation. In the ordered state, our results are in good agreement with $1/S^2$ and Polyakov scaling analyses. Furthermore, we discuss the existence of 'twist waves', and the possibility of a 'twisted spin nematic', two phenomena not anticipated in a conventional large-S approach.

The layout of the paper is as follows. We begin with a gauge-invariant formulation of the Heisenberg model (section 2), and discuss its mean field decoupling in section 3. Next we present the general class of wave functions resulting from our approach (section 4). Spin rotons, quantum exchange gaps and Villain's 'order from disorder' as a simple example of the consequences of spin wave bound-states are discussed in section 5. Section 6 examines the Goldstone mode excitations of quantum helimagnets, demonstrating how triplet spin pairing of magnons associated with a twisted structure results in the formation of an additional *longitudinal* Goldstone mode. The non-linear sigma model for a helimagnet and its biaxial order parameter are presented in section 7; the coefficients for this long-wavelength action are derived in an analogous fashion to Gorkov's calculation of the Landau–Ginzburg coefficients for BCS theory. In section 8 we predict the existence of a twisted spin nematic, one with long-range twist but with no sublattice magnetisation. This spin nematic violates parity but not time reversal symmetry, and has interesting implications for charged systems. We conclude with a brief discussion of future applications of this approach.

2. Gauge-invariant formulation of the Heisenberg model

We begin by rewriting the frustrated Heisenberg model in a form that explicitly displays the rotational gauge invariance associated with continuity of spin flow; in a

Lagrangian formalism, the partition function can be written

$$\begin{aligned} Z &= \int_{\lambda_0}^{\lambda_0+2i\pi T} \mathrm{d}\lambda_j \int \mathcal{D}[b] \exp\left(-\int \mathcal{L}(\tau)\,\mathrm{d}\tau\right) \\ \mathcal{L} &= \mathcal{L}_o + H \qquad \mathcal{L}_o = \sum_j \{b^\dagger{}_j[\partial_\tau - \lambda_j]b_j + 2S\lambda_j\} \end{aligned} \tag{2.1}$$

where H is the Hamiltonian for the frustrated Heisenberg model, written in terms of Schwinger bosons. The fluctuating Onsager potential λ_j imposes the constraints, generating local Onsager cavity fields. Spin indices on the Bose fields have been suppressed ($b^\dagger{}_j = (b^\dagger{}_{j\uparrow}, b^\dagger{}_{j\downarrow})$).

Under an independent rotation of the spin basis at each site

$$\begin{aligned} b^\dagger{}_j &= b_j'^\dagger g_j \\ g_j &= \exp(\tfrac{1}{2}\mathrm{i}\boldsymbol{\theta}_j \cdot \boldsymbol{\sigma}) \end{aligned} \tag{2.2}$$

the spin transforms under the adjoint representation of SU(2)

$$\boldsymbol{S}_j' = \tfrac{1}{2} b_{j\alpha}'^\dagger \boldsymbol{\sigma}_{\alpha\beta} b_{j\beta}' = \exp(-\boldsymbol{\theta}_j \times) \boldsymbol{S}_j . \tag{2.3}$$

The transformed Lagrangian is then $\mathcal{L}^g = \mathcal{L}_o + H^g$, where

$$H^g = \frac{1}{2}\sum J_{ij} \boldsymbol{S}_i \exp(-\boldsymbol{A}_{ij}\times) \boldsymbol{S}_j - \sum_j \boldsymbol{B}_j \cdot \boldsymbol{S}_j . \tag{2.4}$$

The fictitious magnetic field $\boldsymbol{B}_j = \exp(-\boldsymbol{\theta}_j \times)\partial_t \theta_j$ ($\partial_t \equiv \mathrm{i}\partial_\tau$) is induced by the rotation of the spin reference frame. Primes have been dropped for clarity. The exponential $\exp(-\boldsymbol{A}_{ij}\times)$ is a shorthand for the O(3) rotation matrix

$$[\exp(-\boldsymbol{A}_{ij}\times)]^{pq} = [\exp(-\boldsymbol{\theta}_i \times) \exp(\boldsymbol{\theta}_j \times)]^{pq} . \tag{2.5}$$

A smoothly varying gauge transformation $g_j = g(\boldsymbol{R}_j, \tau)$ is locally equivalent to a a twist. For a uniform twist of the co-ordinate axes through an angle $\theta(\boldsymbol{R}) = \boldsymbol{Q} \cdot \boldsymbol{R}$ about the axis $\hat{\boldsymbol{k}}$,

$$g(\boldsymbol{R}) = \exp(\mathrm{i}\boldsymbol{Q} \cdot \boldsymbol{R}\hat{\boldsymbol{k}} \cdot \boldsymbol{\sigma}/2) \tag{2.6}$$

and in this case

$$(\boldsymbol{A}_{ij})_l = \int_j^i \boldsymbol{A}_l \,\mathrm{d}R_l \tag{2.7}$$

where

$$\boldsymbol{A}_l = Q_l \hat{\boldsymbol{k}} \tag{2.8}$$

can be considered to be a spin vector potential. We shall generalise this form to the case of smoothly varying twist fields

$$\boldsymbol{A}_l(\boldsymbol{R}, \tau) = Q_l(\boldsymbol{R}, \tau)\hat{\boldsymbol{k}}(\boldsymbol{R}, \tau). \tag{2.9}$$

The gauge invariant form of our Hamiltonian is then

$$H^g = \frac{1}{2}\sum J_{ij}\boldsymbol{S}_i \exp\left(-\int_j^i \boldsymbol{A}_l\, \mathrm{d}R_l \times\right)\boldsymbol{S}_j - \sum_j \boldsymbol{B}_j \cdot \boldsymbol{S}_j \tag{2.10}$$

Written out explicitly this takes the form

$$\begin{aligned} H^g &= H^g_o + \frac{1}{2}\sum J_{ij}\sin\theta_{ij}\hat{\boldsymbol{k}}_{ij}\cdot(\boldsymbol{S}_i \times \boldsymbol{S}_j) \\ H^g_o &= \frac{1}{2}\sum J_{ij}\{(\boldsymbol{S}_i\cdot\hat{\boldsymbol{k}}_{ij})(\boldsymbol{S}_j\cdot\hat{\boldsymbol{k}}_{ij}) + \cos\theta_{ij}[\boldsymbol{S}_i\cdot\boldsymbol{S}_j - (\boldsymbol{S}_i\cdot\hat{\boldsymbol{k}}_{ij})(\boldsymbol{S}_j\cdot\hat{\boldsymbol{k}}_{ij})]\} \end{aligned} \tag{2.11}$$

where $\theta_{ij}\hat{\boldsymbol{k}}_{ij} = \boldsymbol{A}_{ij}$. Here, $\hat{\boldsymbol{k}}_{ij}$ is the axis of rotation required to rotate co-ordinate system j into co-ordinate system i, and θ_{ij} is the angle of rotation between the two local co-ordinate systems. The Lagrangian $\mathcal{L}^g$ is now explicitly invariant under the local rotational gauge transformation

$$b^\dagger{}_i \rightarrow b^{\dagger'}_i = b^\dagger{}_i g_i \begin{cases} \boldsymbol{B}_j\cdot(\frac{1}{2}\boldsymbol{\sigma}) \rightarrow \boldsymbol{B}_j g^\dagger{}_j\cdot(\frac{1}{2}\boldsymbol{\sigma})g_j - g^\dagger{}_j\partial_\tau g_j \\ \exp\left(-\int_j^i \boldsymbol{A}_l\,\mathrm{d}R_l\times\right) \rightarrow \exp(\boldsymbol{\theta}_i\times)\exp\left(-\int_j^i \boldsymbol{A}_l\,\mathrm{d}R_l\times\right)\exp(-\boldsymbol{\theta}_j\times). \end{cases} \tag{2.12}$$

Expanding the equation for $\boldsymbol{B}'$ and taking spatial derivatives of both sides of the equation for $\boldsymbol{A}_l$, we find that the transformation laws for the magnetic field and spin vector potential take the form

$$\begin{aligned} \boldsymbol{A}_l &\rightarrow \boldsymbol{A}'_l = e^{\boldsymbol{\theta}\times}[\boldsymbol{A}_l - (\nabla_l\boldsymbol{\theta})] \\ \boldsymbol{B} &\rightarrow \boldsymbol{B}' = e^{\boldsymbol{\theta}\times}[\boldsymbol{B} - (\partial_t\boldsymbol{\theta})] \qquad (\partial_t \equiv \mathrm{i}\partial_\tau) \end{aligned} \tag{2.13a}$$

where $t = -\mathrm{i}\tau$ denotes real time. For infinitesimal θ, these transformation laws can be re-written in terms of gauge covariant derivatives

$$\begin{aligned} \boldsymbol{A}_l &\rightarrow \boldsymbol{A}'_l = \boldsymbol{A}_l - \not{\nabla}_l\boldsymbol{\theta} \qquad (\not{\nabla}_l \equiv \nabla_l + \boldsymbol{A}_l\times) \\ \boldsymbol{B} &\rightarrow \boldsymbol{B}' = \boldsymbol{B} - \not{\partial}_t\boldsymbol{\theta} \qquad (\not{\partial}_t \equiv \mathrm{i}\partial_\tau + \boldsymbol{B}\times). \end{aligned} \tag{2.13b}$$

The invariance of the physics under local co-ordinate transformations is directly related to the continuity of spin flow. Since the partition function is independent of the gauge, the variation with respect to the rotation phase $\boldsymbol{\theta}_i$ must vanish, which implies local spin conservation. The variation of the partition function under a local gauge transformation is

$$\delta Z = \int \mathcal{D}[\lambda, b]\exp\left(-\int\mathcal{L}^g(\tau)\,\mathrm{d}\tau\right)\left(\not{\partial}_t\left(\frac{\partial\mathcal{L}^g}{\partial\boldsymbol{B}_i}\right) - \sum_k\frac{\partial\mathcal{L}^g}{\partial\boldsymbol{A}_{ik}}\right)\cdot\delta\boldsymbol{\theta}_i = 0. \tag{2.14}$$

Evaluating the quantity inside the integral, we have

$$\not{\partial}_t(\boldsymbol{S}_i) + \sum_k \boldsymbol{j}_{i\rightarrow k} = 0 \tag{2.15}$$

where the spin current from i to k is

$$\boldsymbol{j}_{i\to k} = J_{ik}[\boldsymbol{S}_i \times \boldsymbol{S}_k^*] \qquad \left(\boldsymbol{S}_k^* = \exp\left(-\int_k^i \boldsymbol{A}_l \, \mathrm{d}R_l \times\right)\boldsymbol{S}_k\right). \tag{2.16}$$

Setting $\boldsymbol{A}_l = 0$, the spin conservation equation (2.14) can be re-written as

$$\partial_t \boldsymbol{S}_i = -\left(\boldsymbol{B} - \sum_k J_{ik}\boldsymbol{S}_k\right) \times \boldsymbol{S}_i \tag{2.17}$$

which is the equation of motion for the precession of a quantum spin. From a quantum fluids perspective, the equation of motion is then equivalent to the the condition of spin conservation.

Finally, if we use (2.10) rather than (2.4), we find

$$\delta Z = \int \mathcal{D}[\lambda, b] \exp\left(-\int \mathcal{L}^g(\tau)\,\mathrm{d}\tau\right)\left(\not{\partial}_t\left(\frac{\partial \mathcal{L}^g}{\partial \boldsymbol{B}(\boldsymbol{R})}\right) + \not{\nabla}_l\left(\frac{\partial \mathcal{L}^g}{\partial \boldsymbol{A}_l(\boldsymbol{R})}\right)\right)\cdot \delta\boldsymbol{\theta}(\boldsymbol{R},\tau) = 0 \tag{2.18}$$

which yields the continuum version of the spin conservation law, in terms of the spin current density $\mathcal{J}_l$

$$\begin{aligned} &\not{\partial}_t \boldsymbol{S}(\boldsymbol{r},t) + \not{\nabla}_l \mathcal{J}_l(\boldsymbol{r},t) = 0 \\ &\mathcal{J}_l(\boldsymbol{r},\tau) = -\frac{\partial \mathcal{L}^g}{\partial \boldsymbol{A}_l} = -\sum_{\boldsymbol{R}} J(\boldsymbol{R})R_l\big[\boldsymbol{S}(\boldsymbol{r}) \times \boldsymbol{S}^*(\boldsymbol{r}-\boldsymbol{R})\big]. \end{aligned} \tag{2.19}$$

3. Mean field decoupling of the Heisenberg model

If we gauge transform the Bose fields $b \to b^g$ inside the path integral, then the Lagrangian becomes $\mathcal{L} \to \mathcal{L}^g$. The integration measure is gauge invariant, $\mathcal{D}[b] = \mathcal{D}[b^g]$, so it follows that the path integral is also gauge invariant

$$Z = Z^g = \int_{\lambda_0}^{\lambda_0+2\mathrm{i}\pi T} \mathrm{d}\lambda_j \int \mathcal{D}[b] \exp\left(-\int \mathcal{L}^g(\tau)\,\mathrm{d}\tau\right). \tag{3.1}$$

It proves convenient to average over gauges, using a normalised weighting function $F(g): \int \mathcal{D}[g]\,F(g) = 1$. We then write $Z = \int \mathrm{d}g\,F(g)Z^g$, or

$$Z = \int_{\lambda_0}^{\lambda_0+2\mathrm{i}\pi T} \mathrm{d}\lambda_j \int \mathcal{D}[g,b]F(g) \exp\left(-\int \mathcal{L}^g(\tau)\,\mathrm{d}\tau\right). \tag{3.2}$$

We shall select the weighting function that simplifies the decoupling procedure. In principle, there are many ways of decoupling the spin interaction; with the exception of collinear magnets, the spontaneous development of a twist $\langle \boldsymbol{S}_i \times \boldsymbol{S}_j \rangle$ implies parity violation, and thus *both* even and odd pairing correlations. A key developmental step in the quantum fluids approach to frustrated magnets is the realisation that *arbitrary* magnet configurations can be described by the action of one, or more twists applied to a ferromagnetic configuration. We shall simplify all calculations by working in a twisted co-ordinate system where the spin correlations are locally ferromagnetic. Formally, this is equivalent to integrating over the spin gauge fields and imposing a gauge fixing condition on the spin configurations

$$\langle \boldsymbol{S}_i^g \times \boldsymbol{S}_j^g \rangle = 0 \tag{3.3}$$

The transformed Lagrangian now becomes

$$\mathcal{L}^g = \mathcal{L}_o^g + H_o^g + \sum (J_{ij} \sin\theta_{ij} \hat{\boldsymbol{k}}_{ij} - \boldsymbol{\lambda}_{ij}) \cdot (\boldsymbol{S}_i \times \boldsymbol{S}_j) \tag{3.4}$$

where an integral over the Lagrange multipliers $\boldsymbol{\lambda}_{ij}$ fixes the average gauge; in this frame of reference the twist vanishes and locally ferromagnetic *even* parity pairing results in both the particle-hole and Cooper channels. Thus in the twisted reference frame the magnet can be treated as an even parity, triplet paired Bose fluid.

We shall restrict our attention to the cases where the equilibrium magnetic structure is uniformly twisted about an axis $\hat{\boldsymbol{k}}$

$$\boldsymbol{S}_j = \exp[(\boldsymbol{Q}\cdot\boldsymbol{R}_j)\hat{\boldsymbol{k}}\times]\boldsymbol{S}'_j \begin{cases} g_j = \exp(\frac{1}{2}\boldsymbol{\theta}_j \cdot \boldsymbol{\sigma}) \\ \boldsymbol{\theta}_j = (\boldsymbol{Q}\cdot\boldsymbol{R})\hat{\boldsymbol{k}}. \end{cases} \tag{3.5}$$

The special case of collinear antiferromagnets is recovered when $2\boldsymbol{Q} \equiv 0$. We now introduce the triplet Cooper and the singlet particle-hole pairing fields

$$\begin{aligned} \boldsymbol{B}_{\boldsymbol{q}}^\dagger &= \mathrm{i} b^\dagger{}_{\boldsymbol{q}\alpha} (\boldsymbol{\sigma}\sigma^2)_{\alpha\beta} b^\dagger{}_{-\boldsymbol{q}\beta} \\ D_{\boldsymbol{q}}^\dagger &= b^\dagger{}_{\boldsymbol{q}\uparrow} b_{\boldsymbol{q}\uparrow} + b_{\boldsymbol{q}\downarrow} b^\dagger{}_{\boldsymbol{q}\downarrow} \end{aligned} \tag{3.6}$$

and make the pairing *ansatz*

$$\begin{aligned} \langle \boldsymbol{B}_{\boldsymbol{q}}^\dagger \rangle &= 2\eta_{\boldsymbol{q}}\, \hat{\boldsymbol{k}} \\ \langle D_{\boldsymbol{q}}^\dagger \rangle &= 2\alpha_{\boldsymbol{q}} \end{aligned} \tag{3.7}$$

where $\hat{\boldsymbol{k}}$ is the twist axis and $\eta_{\boldsymbol{q}}$ and $\alpha_{\boldsymbol{q}}$ are even functions of $\boldsymbol{q}$. In the classical limit $S \to \infty$, these pairing correlations become $\alpha_{\boldsymbol{q}} = \eta_{\boldsymbol{q}} = S\delta_{\boldsymbol{q}} + \mathrm{O}(1)$.

The terms in the Hamiltonian containing $\boldsymbol{S}_i \times \boldsymbol{S}_j$ violate parity and will mix even and odd parity pairs. For stationarity with respect to λ_{ij}, these terms must vanish at the saddle point

$$\boldsymbol{\lambda}_{ij} = J_{ij} \sin\theta_{ij} \hat{\boldsymbol{k}}_{ij}. \tag{3.8}$$

At the saddle point, the twisted Hamiltonian is then

$$H = \frac{1}{2}\sum_{ij} J_{ij} \{\gamma^{(+)} \boldsymbol{S}_i \cdot \boldsymbol{S}_j + \gamma^{(-)}[\boldsymbol{S}_i \cdot \boldsymbol{S}_j - 2(\boldsymbol{S}_i \cdot \hat{\boldsymbol{k}})(\boldsymbol{S}_j \cdot \hat{\boldsymbol{k}})]\} \tag{3.9}$$

where $\gamma^{(\pm)} = \frac{1}{2}(1 \pm \cos(\boldsymbol{Q}\cdot\boldsymbol{R}_{ij}))$. The terms in the Hamiltonian can now be decoupled into even parity pairs

$$\begin{aligned} \boldsymbol{S}_i \cdot \boldsymbol{S}_j &= \tfrac{1}{2} : D^{\dagger}{}_{ij} D_{ij} : - S^2 \\ \boldsymbol{S}_i \cdot \boldsymbol{S}_j - 2(\boldsymbol{S}_i \cdot \mathrm{k})(\boldsymbol{S}_j \cdot \hat{\boldsymbol{k}}) &= \tfrac{1}{2} B^{(t)\dagger}_{ij} B^{(t)}_{ij} - S^2 \end{aligned} \tag{3.10}$$

where $B^{(t)\dagger}_{ij} = \hat{\boldsymbol{k}} \cdot \boldsymbol{B}^{\dagger}{}_{ij}$. Restricting our attention to zero-momentum pairing we can write the coupling as a BCS Hamiltonian

$$H_{\mathrm{BCS}} = (1/4) \sum_{\boldsymbol{q}\,\boldsymbol{q}'} [\mathcal{J}^{+}_{\boldsymbol{q}\,\boldsymbol{q}'} D_{\boldsymbol{q}}{}^{\dagger} D_{\boldsymbol{q}'} - \mathcal{J}^{-}_{\boldsymbol{q}\,\boldsymbol{q}'} B^{(t)\dagger}_{\boldsymbol{q}} B^{(t)}_{\boldsymbol{q}'}] - N J(\boldsymbol{Q}) S^2/2. \tag{3.11}$$

Here N is the number of sites and the pairing potentials are

$$\mathcal{J}^{\pm}_{\boldsymbol{q}\boldsymbol{q}'} = \tfrac{1}{2}\{J(\boldsymbol{q}+\boldsymbol{q}') \pm \tfrac{1}{2}[J(\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{Q}) + J(\boldsymbol{q}+\boldsymbol{q}'-\boldsymbol{Q})]\}_S. \tag{3.12}$$

The subscript S denotes symmetrisation with respect to q and q'. Here, the first and second terms are the pure ferromagnetic and antiferromagnetic pairing potentials respectively. Note that the constraint has played a role in deriving this symmetric pairing Hamiltonian. The same mean field theory can also be obtained from a naïve Hartree-Fock decoupling, of the Hamiltonian

$$H' = H + \frac{1}{2} \sum_{ij} J_{ij} \cos(\boldsymbol{Q}\cdot\boldsymbol{R}_{ij})(n_i n_j/4 - S^2). \tag{3.13}$$

The second term is zero under the constraint, but generates a more symmetric mean field decoupling of the problem. The resulting mean field Hamiltonian is

$$\begin{aligned} H_{\mathrm{MF}} = \sum_{\boldsymbol{q}} \{ (h_{\boldsymbol{q}} - \lambda)[b^{\dagger}{}_{\boldsymbol{q}\uparrow} b_{\boldsymbol{q}\uparrow} + b_{-\boldsymbol{q}\downarrow} b^{\dagger}{}_{-\boldsymbol{q}\downarrow}] - [\Delta_{\boldsymbol{q}} b^{\dagger}{}_{\boldsymbol{q}\uparrow} b^{\dagger}{}_{-\boldsymbol{q}\downarrow} + \mathrm{HC}]\} \\ + E_c + 2N\lambda(S + \tfrac{1}{2}) \end{aligned} \tag{3.14}$$

where we chose $\hat{\boldsymbol{k}} = \hat{\boldsymbol{z}}$, and

$$E_c = \sum_{\boldsymbol{q}\,\boldsymbol{q}'} \{ \Delta_{\boldsymbol{q}} [\mathcal{J}^{-}]^{-1}_{\boldsymbol{q}\,\boldsymbol{q}'} \Delta_{\boldsymbol{q}'} - h_{\boldsymbol{q}} [\mathcal{J}^{+}]^{-1}_{\boldsymbol{q}\,\boldsymbol{q}'} h'_{\boldsymbol{q}} \} - N J(\boldsymbol{Q}) S^2/2 \tag{3.15}$$

is the spin condensate energy. The quasiparticle energies of H_{MF} are $\omega_{\boldsymbol{q}} = [(\tilde{h}_{\boldsymbol{q}})^2 - \Delta_{\boldsymbol{q}^2}]^{1/2}$, where $\tilde{h}_{\boldsymbol{q}} = h_{\boldsymbol{q}} - \lambda$, so the total mean field free energy per unit cell is

$$F = \sum_{\boldsymbol{q}} 2T \ln[2\sinh(\beta\omega_{\boldsymbol{q}}/2)] + E_c + N\lambda(2S+1). \tag{3.16}$$

Differentiating with respect to $h_{\boldsymbol{q}}$, $\Delta_{\boldsymbol{q}}$, and λ yields

$$\begin{aligned} h_{\boldsymbol{q}} &= \int_{\boldsymbol{q}}' \mathcal{J}^{+}_{\boldsymbol{q}\boldsymbol{q}'} \alpha'_{\boldsymbol{q}} \\ \Delta_{\boldsymbol{q}} &= \int_{\boldsymbol{q}}' \mathcal{J}^{-}_{\boldsymbol{q}\boldsymbol{q}'} \eta'_{\boldsymbol{q}} \\ S + \tfrac{1}{2} &= \int_{\boldsymbol{q}} \alpha_{\boldsymbol{q}} \end{aligned} \tag{3.17}$$

where $\int_q \equiv \int \mathrm{d}^2q/(2\pi)^2$, and

$$(2\alpha_q,\ 2\eta_q) = (\langle D^\dagger{}_q\rangle, \langle B_q^{(t)\dagger}\rangle) = [\coth(\beta\omega_q/2)/\omega_q](\tilde{h}_q, \Delta_q). \tag{3.18}$$

The last expression in (3.17) is the mean field constraint. For ferromagnets ($Q = 0$), the antiferromagnetic pairing potential $\mathcal{J}^-$ vanishes, and these equations revert to Takahashi's equations for ferromagnets [8]. For bipartite antiferromagnets, the ferromagnetic pairing potential $\mathcal{J}^+$ vanishes, and the second equation reverts to the Arovas–Auerbach [6, 7, 21] result.

Finally, in order to determine $\boldsymbol{Q}$, we differentiate the free energy with respect to $\boldsymbol{Q}$

$$\int_{q\,q'} [\alpha_q \nabla_Q \mathcal{J}^+_{q\,q'} \alpha'_q - \eta_q \nabla_Q \mathcal{J}^-_{q\,q'} \eta'_q] - \tfrac{1}{2}\nabla_Q J(Q) S^2 = 0. \tag{3.19}$$

At zero temperature and large-S, the bosons condense at $q = 0$, and there is a pole in the occupation functions $\alpha_q \sim \eta_q \sim S^* \delta_q$ corresponding to a finite magnetisation S^*. As $S \to \infty$, the pole dominates, and $S^*/S \to 1$, so

$$\begin{aligned} h_q &= S\mathcal{J}^+_{q\,0} \\ \Delta_q &= S\mathcal{J}^-_{q\,0}. \end{aligned} \tag{3.20}$$

λ takes the smallest value consistent with $\omega_0 = 0$, which gives $\lambda = SJ(\boldsymbol{Q})$. Thus the dispersion predicted in the large-S limit is

$$\omega_q^2 = S^2[J(q) - J(Q)][\tfrac{1}{2}(J(q+Q) + J(q-Q)) - J(Q)] \tag{3.21}$$

which is exactly the spin wave spectrum of the twisted magnet [24]. The stability equation (3.19) becomes simply $\nabla_Q J(\boldsymbol{Q}) = 0$, selecting allowed values of $\boldsymbol{Q}$ in the large-S limit.

4. Ground-state wavefunction: relationship with RVB

Let us briefly examine the type of ground state wave function furnished by this approach. The Bogoliubov quasiparticles for the mean field Hamiltonian have the form

$$\begin{aligned} a^\dagger{}_{q\sigma} &= u_q b^\dagger_{q\sigma} - v_q b_{-q-\sigma} \\ u_q^2 &= \tfrac{1}{2}[(\tilde{h}_q/\omega_q) + 1] \qquad u_q v_q = (\Delta_q/2\omega_q). \end{aligned} \tag{4.1}$$

The mean field ground state wavefunction that is annihilated by the quasiparticles has the Jastrow form

$$|\Psi\rangle = \exp\left\{\sum_q f_q b^\dagger_{q\uparrow} b^\dagger{}_{-q\downarrow}\right\}|0\rangle \tag{4.2}$$

where $f_q = (v_q/u_q)$. In the thermodynamic limit, $N \to \infty$, $f_o \to 1$ and the ground state develops an infinite accumulation of particles in the $q = 0$ state: $\langle b^\dagger{}_o\rangle \to \sqrt{NS^*}$, permitting us to divide the wavefunction into a normal fluid and condensate

$$|\Psi\rangle = |\Psi_N\rangle|\Psi_C\rangle \tag{4.3}$$

where

$$
\begin{aligned}
|\Psi_N\rangle &= \exp\left\{\sum_{q\neq 0} f_q b^\dagger{}_{q\uparrow} b^\dagger{}_{-q\downarrow}\right\}|0\rangle \\
|\Psi_C\rangle &= \exp\left\{\sqrt{NS^*}[b^\dagger{}_{0\uparrow} + b^\dagger{}_{0\downarrow}]\right\}|0\rangle.
\end{aligned}
\tag{4.4}
$$

In the untwisted reference frame, the fully constrained spin wavefunction is then

$$
|\tilde{\Psi}\rangle = P_S g^\dagger |\Psi_N\rangle|\Psi_C\rangle = P_S|\tilde{\Psi}_N\rangle|\tilde{\Psi}_C\rangle \tag{4.5}
$$

where $g^\dagger = \exp\left[-\mathrm{i}\sum_j \boldsymbol{Q}\cdot\boldsymbol{R}_j S_j^z\right]$ twists the wavefunction and P_S projects the Gibbs ensemble with $2S$ spin quanta per site. If we now write the Bose fields in terms of the untwisted creation operators, denoted here by $\tilde{b}^\dagger{}_{q\sigma}$

$$
\tilde{b}^\dagger_{q\sigma} = \sum_{q} \tilde{b}^\dagger_{j\sigma}\exp[\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{R}_j)] = \sum_j b^\dagger{}_{j\sigma}\exp[\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{R}_j - \theta_j\sigma/2)] = b^\dagger{}_{q-Q\sigma/2,\sigma} \tag{4.6}
$$

then the twisted normal fluid and condensate are

$$
\begin{aligned}
|\tilde{\Psi}_N\rangle &= \exp\left\{\frac{1}{2}\sum_{q\neq 0} f_q^{(+)}\hat{\boldsymbol{k}}\cdot\boldsymbol{B}^\dagger{}_q - i f_q^{(-)} B^\dagger{}_q\right\}|0\rangle \\
|\tilde{\Psi}_C\rangle &= \exp\left\{\sqrt{NS}[\tilde{b}^\dagger{}_{Q/2\uparrow} + \tilde{b}^\dagger{}_{-Q/2\downarrow}]\right\}|0\rangle.
\end{aligned}
\tag{4.7}
$$

Here $f_q^{(\pm)} = \frac{1}{2}[f_{q+Q/2} \pm f_{q-Q/2}]$ and $\tilde{B}^\dagger{}_q = -\tilde{b}^\dagger{}_q\sigma^2\tilde{b}^\dagger{}_{-q}$. The coexistence of even and odd parity pairing in this wavefunction generates the twist. In real space, this state can be written as an RVB wavefunction

$$
|\tilde{\Psi}\rangle = P_S\exp\left\{\sum_{ij} f(\boldsymbol{R}_{ij})[\cos(\boldsymbol{Q}\cdot\boldsymbol{R}_{ij}/2)\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{B}}^\dagger{}_{ij} + \sin(\boldsymbol{Q}\cdot\boldsymbol{R}_{ij}/2)\tilde{B}^\dagger{}_{ij}\right\}|0\rangle \tag{4.8}
$$

where the pairing wavefunction is $f(\boldsymbol{R}) = \sum_q f_q \cos\boldsymbol{q}\cdot\boldsymbol{R}$. This is a generalisation of the Jastrow wavefunction considered for bipartite lattices [6, 5].

In the special cases where the magnet is collinear, $f_q = -f_{q+Q}$, so $f_q^{(+)} = 0$ and triplet pairing *vanishes* in a collinear magnet. The normal state is accordingly an isotropic singlet. In general, the static spin correlations associated with this ground state are readily evaluated by re-expressing the spin operators in terms of the quasi-particle operators. In the twisted reference frame, the spin operators take the form $\boldsymbol{S}(\boldsymbol{R}) = \sum \boldsymbol{S}_q \exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{R})$, where

$$
\begin{aligned}
S^+(q) &= \frac{1}{2}\sum_k (u_+v_- + u_-v_+)[a^\dagger{}_{k-\uparrow}a^\dagger{}_{-k+\uparrow} + a_{-k-\downarrow}a_{k+\downarrow}] + \mathrm{PH} \\
S^3(q) &= \frac{1}{2}\sum_k (u_+v_- - u_-v_+)[a^\dagger{}_{k-\uparrow}a^\dagger{}_{-k+\downarrow} + a_{-k-\uparrow}a_{k+\downarrow}] + \mathrm{PH}
\end{aligned}
\tag{4.9}
$$

Here $S^{\pm} = S^x \pm \mathrm{i}S^y$, $\boldsymbol{k}^{\pm} = \boldsymbol{k} \pm \boldsymbol{q}/2$, $(u_{\pm}, v_{\pm}) = (u_{\boldsymbol{k}^{\pm}}, v_{\boldsymbol{k}^{\pm}})$ and the quasiparticle-hole terms denoted by PH have been dropped, since they annihilate the ground state. Evaluating the expectation value of the static spin correlations in the ground state yields

$$\langle S^a(\boldsymbol{q})S^a(\boldsymbol{q})\rangle = \begin{cases} \chi^+(\boldsymbol{q}) & (a = 1, 2) \\ \chi^-(\boldsymbol{q}) & (a = 3) \end{cases} \tag{4.10}$$

where spin components are measured in the twisted reference frame and $\chi^{\pm}(\boldsymbol{q}) = \frac{1}{4}\sum_{\boldsymbol{q}}(u_+v_{\pm} - u_-v_+)^2$. By re-writing the coherence factors in terms of $\alpha_{\boldsymbol{q}} = (u_{\boldsymbol{q}}^2 - \frac{1}{2})$ and $\eta_{\boldsymbol{q}} = u_{\boldsymbol{q}}v_{\boldsymbol{q}}$ we find that in real space

$$\chi^{\pm}(\boldsymbol{R}) = \tfrac{1}{2}(\alpha(\boldsymbol{R})^2 \pm \eta(\boldsymbol{R})^2) \tag{4.11}$$

where

$$(\alpha(\boldsymbol{R}), \eta(\boldsymbol{R})) = \sum_{\boldsymbol{q}\neq 0}(\alpha_{\boldsymbol{q}}, \eta_{\boldsymbol{q}}) \cos \boldsymbol{q}\cdot\boldsymbol{R}. \tag{4.12}$$

Transforming back to the untwisted reference frame, the static spin correlations are

$$\langle S^a(x)S^b(x')\rangle = \chi^+(\boldsymbol{R}) \begin{bmatrix} \cos \boldsymbol{Q}\cdot\boldsymbol{R} & -\sin \boldsymbol{Q}\cdot\boldsymbol{R} & \\ \sin \boldsymbol{Q}\cdot\boldsymbol{R} & \cos \boldsymbol{Q}\cdot\boldsymbol{R} & \\ & & 0 \end{bmatrix} + \chi^-(\boldsymbol{R}) \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} + \langle S^a(x)S^b(x')\rangle_{\mathrm{c}}. \tag{4.13}$$

Here $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}'$, and we have separated out the condensate component

$$\langle S^a(x)S^b(x')\rangle_{\mathrm{c}} = [S^*]^2 \begin{bmatrix} cc' & sc' & \\ cs' & ss' & \\ & & 0 \end{bmatrix} \tag{4.14}$$

where $(c, s) = (\cos \boldsymbol{Q}\cdot\boldsymbol{x}, \sin \boldsymbol{Q}\cdot\boldsymbol{x})$ and $(c', s') = (\cos \boldsymbol{Q}\cdot\boldsymbol{x}', \sin \boldsymbol{Q}\cdot\boldsymbol{x}')$. For a collinear magnet, which is a singlet in the absence of the spin condensate, $[\alpha^2(\boldsymbol{R}) + \eta^2(\boldsymbol{R})] = [\alpha^2(\boldsymbol{R}) - \eta^2(\boldsymbol{R})] \cos \boldsymbol{Q}\cdot\boldsymbol{R}$, and the spin correlations are isotropic, apart from the uniaxial component derived from the condensate. By contrast, if the magnet is non-collinear, i.e. $2\boldsymbol{Q} \not\equiv 0$, then the zero-point fluctuations are triplet paired, and the normal fluid exhibits uniaxial order defined by the axis of the twist. The combined system of condensate plus anisotropically paired spin fluid possesses *biaxial order* defined by the sublattice magnetisation and the twist.

It is illuminating to examine how the presence or absence of a sublattice magnetisation depends on the spin pairing. At zero temperature, we may rewrite the constraint equation as

$$S = S^* + \int_{\boldsymbol{q}} v_{\boldsymbol{q}}^2 = S^* + \int_{\boldsymbol{q}} f_{\boldsymbol{q}}^2[1 - f_{\boldsymbol{q}}^2]^{-1}. \tag{4.15}$$

A sublattice magnetisation will in general occur for sufficiently large S, provided that the integral on the right hand side is finite. In general, it is not necessary to have infinite range bonds, and the higher the dimension, the more readily short range pairing can give rise to an infinite range sublattice magnetisation. As an example, consider a nearest neighbour RVB wavefunction for a square 2D lattice of the form first considered by Sutherland. Here, for $S = \frac{1}{2}$

$$|\psi\rangle = P_{1/2} \sum_{\{i,j\}} \prod_{(i,j)} h(\mathbf{R}_{ij}) B^{\dagger}{}_{ij} |0\rangle \tag{4.16}$$

where the sum is over all possible bond configurations $\{i, j\}$ and $h(\mathbf{R}) = \sin(\mathbf{Q}.\mathbf{R}/2)$, $\mathbf{Q} = (\pi, \pi)$ for nearest neighbour bonds, but vanishes otherwise. In this state triplet pairing vanishes. Putting $h(\mathbf{R}) = f(\mathbf{R}) \sin(\mathbf{Q} \cdot \mathbf{R}/2)$, then, up to a normalisation

$$f_{\mathbf{q}} = \tfrac{1}{2} f(\cos q_x + \cos q_y) \qquad \text{Sutherland RVB.} \tag{4.17}$$

In the Jastrow generalisation of this wavefunction to higher S, normalisation of $f_{\mathbf{q}}$ is achieved from condition (4.15). Since $f_{\mathbf{q}}^2 \sim f^2(1 - q^2)$, as $f \to 1$, the fluctuation integral in (4.15) diverges in two dimensions $\sim ln[1/(1 - f^2)]$, and an arbitrarily large value of S can be achieved without developing a condensate at $\mathbf{q} = 0$. From this argument it follows that the Sutherland state, and its generalisation to large-S will always be disordered. Clearly, phase space and the form of the RVB wavefunction *in momentum space* plays a strong role in determining whether there is a magnetic condensate. For a three dimensional Sutherland State, the phase space integral that determines S is convergent ($S_c \approx 1.57$), even when $f_{q/o} = 1$, suggesting that for larger values of $S > S_c$, the 3D Sutherland wavefunction will exhibit long-range magnetic order.

5. 'Order from disorder'

Néel antiferromagnets are a very special class of spin structure, where the magnetic vector $\mathbf{Q}$ characterising the long-range order $\langle \mathbf{S}(\mathbf{R}) \cdot \mathbf{S}(0) \rangle \sim S^2 \cos \mathbf{Q} \cdot \mathbf{R}$ lies at the zone centre of the Brillouin zone. More generally however, the introduction of frustration into the Heisenberg model

$$\begin{aligned} H &= \tfrac{1}{2} \sum_{ij} J(\mathbf{R}_{ij}) \mathbf{S}_i \cdot \mathbf{S}_j \\ J(q) &= 2J_1(c_x + c_y) + 4J_2(c_x c_y) + 2J_3(c_{2x} + c_{2y}) + \cdots \end{aligned} \tag{5.1}$$

forces the $\mathbf{Q}$ vector to a point of lower symmetry. In this case, the ground-state violates both the spin rotation symmetry and the lattice rotational point-group symmetry. Whereas the latter can be understood in terms of a classical picture of magnetism, the break-down of discrete lattice symmetry is actually driven by spin fluctuations in the normal fluid. This phenomenon was first studied by Villain [25, 26], who called it 'order from disorder'.

Consider the class of helimagnetic structures [27,28], where the spatial precession of the magnetisation $\mathbf{M}(\mathbf{x})$ defines a local SO(3) co-ordinate basis $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$, with

$$\mathbf{M}(\mathbf{x}) = S\hat{e}_1(\mathbf{x}) \qquad \nabla_l \mathbf{M}(\mathbf{x}) = Q_l \hat{e}_3 \times \mathbf{M}(\mathbf{x}) \qquad \hat{e}_1 = [\cos(\mathbf{Q} \cdot \mathbf{x})\hat{\imath} + \sin(\mathbf{Q} \cdot \mathbf{x})\hat{\jmath}]. \tag{5.2}$$

Classically, this state has energy $E_o = NS^2J(\boldsymbol{Q})/2$, and is favored when $\nabla_Q J(\boldsymbol{Q}) = 0$ away from the zone centre. For instance, in the frustrated Heisenberg model, for small J_3,when $|J_1 - 2J_2| < 4J_3$, a helimagnetic state with $\boldsymbol{Q} = (\pi, \theta)$ or $(\pi - \theta, \pi)$ forms. The spinwave spectrum for a helimagnet is

$$\omega_{\boldsymbol{q}}^2 = (S)^2[J(\boldsymbol{q}) - J(\boldsymbol{Q})][\tfrac{1}{2}(J(\boldsymbol{q}+\boldsymbol{Q}) + J(\boldsymbol{q}-\boldsymbol{Q})) - J(\boldsymbol{Q})]. \tag{5.3}$$

In helimagnets there is no axial symmetry: spin rotation symmetry is fully broken, and action of infinitesimal rotation operators on the spin condensate gives rise to three associated Goldstone zero-modes at $\boldsymbol{q} = \pm(\boldsymbol{Q}, 0)$. In the absence of spin fluctuations, the environment surrounding the spin condensate has the full crystal point symmetry, and the Goldstone mode structure reflects this symmetry. Since $\boldsymbol{Q}$ is no longer at the zone centre, the spectrum also contains two additional zero-modes at $\boldsymbol{q} = \pm\boldsymbol{Q}' = \pm(Q_y, Q_x)$. Physically, these modes correspond to the application of a twist about an axis perpendicular to the first, at wavevector $\boldsymbol{Q} \pm \boldsymbol{Q}'$. These modes are O(3) phasons: zero modes that distort the structure continuously by redistributing the exchange energy amongst the bonds. Fluctuations, both quantum and thermal, remove the phason zeromodes. Zero-point and thermal fluctuations are minimised in the structure with maximum ferromagnetic spin alignment, stabilising the helimagnet against the development of a double twist. Spin fluctuations sense the $\boldsymbol{Q}$ vector and generate a weak violation of lattice symmetry in the environment surrounding the classical spin condensate. Consequently , new gaps appear in the spin wave spectrum, leading to roton-like minima in the spin wave dispersion [22] (see figure 2).

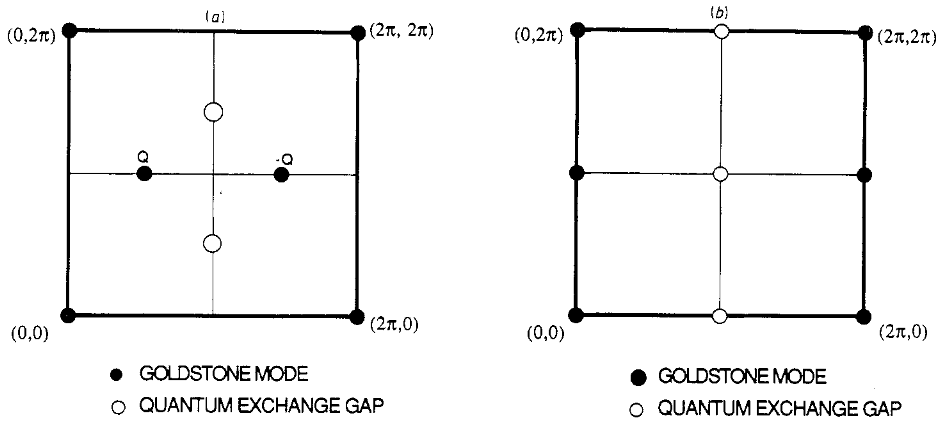

Figure 2. Location of Goldstone modes and quantum exchange gaps for (a) a helimagnet ($\boldsymbol{Q} = (\pi - \theta, \pi)$) and ($b$) a collinear magnet ($\boldsymbol{Q} = (0, \pi)$) in momentum space (twisted reference frame).

To see this effect in action in the quantum fluids approach, we reintroduce the residual zero-point fluctuations from the large-S spectrum into the right-side of the pairing equations and iterate once. From the constraint equation, the magnetisation pole is renormalised by the fluctuations $S^* = S - S_c$, where $S_c + \frac{1}{2} = \sum_{\boldsymbol{q}} \alpha_{\boldsymbol{q}}$, as in spin wave theory [29,23] (figure 3.). The shift $\delta\lambda$ in λ is adjusted to preserve the Goldstone mode at $\boldsymbol{q} = 0$ which gives, at zero temperature

$$\lambda = J(\boldsymbol{Q})[S + \tfrac{1}{2}] + \frac{1}{2S}\int_{\boldsymbol{q}} \omega_{\boldsymbol{q}}. \tag{5.4}$$

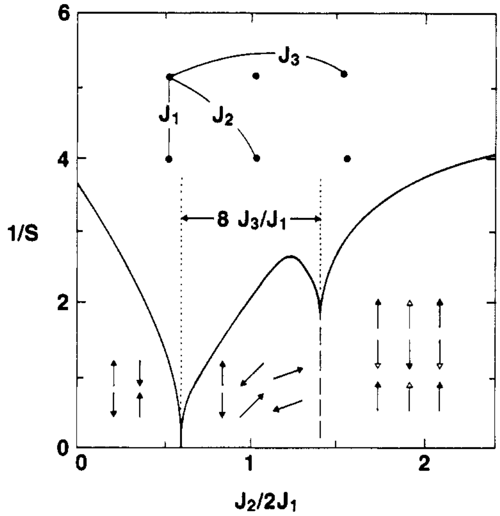

Figure 3. Phase diagram obtained by comparison of the classical sublattice magnetisation and the first quantum corrections, evaluated for $J_3/J_1 = 0.1$. Insets: classical spin configurations.

If we interpret λ as the Onsager reaction field, then the first term is the Curie–Weiss exchange field; the second term is derived from the reaction of the normal fluid, and is related to the zero-point energy per site, as found by Brout and Thomas [18].

A simple example of 'order from disorder' occurs in the case of large diagonal coupling $J_1/2J_2 = \epsilon < 1$, where the minimum of $J(\boldsymbol{Q})$ is at $\boldsymbol{Q} = (0, \pi)$ or $(\pi, 0)$. Classically, the state behaves as two interpenetrating Néel sublattices that can be independently rotated. This is the phason mode. The spin wave spectrum has rotational Goldstone modes at $\boldsymbol{q} = 0, \boldsymbol{Q}$ and phason modes at, $\boldsymbol{Q}' = (Q_y, -Q_x)$ and (π, π). After the first iteration in the pairing equations, the spectrum remains gapless at $\boldsymbol{q} = 0, \boldsymbol{Q}$, but acquires a quantum exchange gap Δ_1 at $\boldsymbol{Q}^* = \boldsymbol{Q}', (\pi, \pi)$, giving rise to a roton minima in the spectrum of form $\omega_{\boldsymbol{q}+\boldsymbol{Q}^*} = [\Delta_1^2 + (c_x q_x)^2 + (c_y q_y)^2]^{1/2}$. Taking $J_3 = 0$, $\boldsymbol{Q} = (0, \pi)$, corresponding to ferromagnetic correlations along the x axis, the pairing fields are

$$\begin{aligned} \mathcal{J}^{(+)}_{\boldsymbol{qq}'} &= 4J_2 \epsilon c_x c'_x \\ \mathcal{J}^{(-)}_{\boldsymbol{qq}'} &= 4J_2[c_x c_y c_{x'} c_{y'} + \epsilon c_y c'_y] \end{aligned} \tag{5.5}$$

After the first iteration of the pairing equations, the quantum exchange gap at $\boldsymbol{Q}^* = (\pi, 0)$ and (π, π) is

$$(\Delta_1)^2 = 2[\tilde{h}_{\boldsymbol{Q}^*} \delta \tilde{h}_{\boldsymbol{Q}^*} - \Delta_{\boldsymbol{Q}^*} \delta \Delta_{\boldsymbol{Q}^*}] = 32\epsilon(1-\epsilon) S (J_2)^2 \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \phi(\boldsymbol{q}) \tag{5.6}$$

where

$$\phi(\boldsymbol{q}) = \frac{(c_y^2 - c_x^2)\epsilon + c_x(c_y^2 - 1)}{\sqrt{(1+\epsilon c_x)^2 - (c_x c_y + \epsilon c_y)^2}} + \mathrm{O}(1). \tag{5.7}$$

The same result can be obtained in spin wave theory from spin wave interactions [22]. For small ϵ,

$$(\Delta_1)^2 = 2S(J_1)^2 \int \frac{\mathrm{d}^2 q}{(\pi)^2} \frac{(\frac{1}{2}[c_x^2 + c_y^2][1 + c_x^2 c_y^2] - 2c_x^2 c_y^2)}{(1 - (c_x c_y)^2)^{3/2}} = 4.16S(J_1)^2$$

which was also recently derived in the context of a sigma model analysis of this problem [30]. This gap stabilises the collinear phase, and at distances $l > l_o = c/\Delta_1$ ($c \sim \sqrt{c_x c_y}$) the two sublattices become locked together by the short wavelength fluctuations, breaking the Z_4 lattice symmetry to Z_2. This transition can be regarded as a 'spin binding' transition in which the effective spins of the moments are doubled.

The basic character of this transition is set by the constraint equation, which self-consistently determines the temperature dependent quantum exchange gap. In the large-S limit, the spin fluctuations accumulate in the vicinity of the Goldstone modes and the quantum exchange gaps. If we approximate the spectrum in the vicinity of these points by

$$\omega_{\boldsymbol{q}}^2 = \begin{cases} c^2(\boldsymbol{q} - \boldsymbol{Q}_i)^2 + \Delta_o^2 & (\boldsymbol{Q}_i = 0, \boldsymbol{Q}) \\ c^2(\boldsymbol{q} - \boldsymbol{Q}_i)^2 + \Delta_o^2 + \Delta_1^2 & (\boldsymbol{Q}_i = \boldsymbol{Q}^*, (\pi, \pi)) \end{cases} \tag{5.8}$$

where

$$\begin{aligned} \Delta_o^2 &= c^2/\xi(T)^2 = \tilde{h}(\boldsymbol{q} = 0)^2 - \Delta(\boldsymbol{q} = 0)^2 \\ (\Delta_1)^2 &= 2[\tilde{h}_{\boldsymbol{Q}^*} \delta\tilde{h}_{\boldsymbol{Q}^*} - \Delta_{\boldsymbol{Q}^*} \delta\Delta_{\boldsymbol{Q}^*}] = c^2/l_o^2 \end{aligned} \tag{5.9}$$

and $\delta\lambda_{\boldsymbol{q}}, \delta\Delta_{\boldsymbol{q}}, \delta h_{\boldsymbol{q}}$ are the deviations from the zero temperature, large-S values of these quantities. Here, we have assumed for clarity, an isotropic spin wave velocity $c = 4J_2Sa$, valid in the limit of small J_1. In the large-S limit, the finite temperature constraint equation is then

$$S + \frac{1}{2} = \frac{1}{8\pi J_2 S} \left[\int_{\Delta_o}^{2\pi c/a} + \int_{\sqrt{\Delta_o^2 + \Delta_1^2}}^{2\pi c/a} \right] \mathrm{d}x \, \coth\left[\frac{\beta x}{2}\right] \tag{5.10}$$

where the change of variables $x = c|\boldsymbol{q} - \boldsymbol{Q}_i|$ has been made and a cut-off has been imposed on the momentum integrals the neighbourhood of the Goldstone and phason modes. At high temperatures where $\Delta_1 = 0$, the spin bosons accumulate equally near both minima in the excitation spectra. Carrying out the frequency integrals, the high temperature behaviour of the spin correlation length $\xi = c/\Delta_o$ is

$$\xi = a \exp(2\pi/g) \tag{5.11}$$

with coupling constant

$$\frac{1}{g} = \frac{J_2 S^2}{T} - \frac{1}{2\pi} \ln\left(\frac{2Ta}{c}\right) + \mathrm{O}(S). \tag{5.12}$$

This result reproduces the one-loop scaling equations for the 2D Heisenberg antiferromagnet derived by Chakravarty *et al* [31]. At low temperatures, once $\xi(T) \sim l_o$, the logarithmic divergence associated with the spin roton modes is cut-off by the quantum exchange gap. The spin concentration is now concentrated solely in the region of the Goldstone modes, effectively doubling the spin S of the system. The low temperature behaviour of the spin correlation length is then

$$\xi(T) = l_o \exp(2\pi/g^*) \tag{5.13a}$$

where the renormalised coupling constant is given by

$$\frac{2\pi}{g^*} = 4\pi\left[\frac{1}{g} - \frac{1}{2\pi}\ln\left(\frac{l_o}{a}\right)\right]. \tag{5.13b}$$

The factor of two difference in the exponent for ξ results from the freezing of the out-of-phase 'phason' modes, locking the two Néel sublattices together on length scales greater than l_o. This results in the replacement

$$J_2S^2 \to 2J_2S^2 \tag{5.14}$$

effectively halving of the coupling constant. The additional logarithmic correction accounts for the renormalisation of the coupling constant by fluctuations with wavelengths between a and l_o.

These results essentially reproduce the results of scaling theory [30]. The transition from the high-temperature to the low-temperature regime is accompanied by an Ising phase transition associated with the breaking of the discrete lattice rotation symmetry from Z_4 to Z_2. The associated soft Ising variable is

$$\sigma(\boldsymbol{R}) = \frac{1}{4S^2}[\boldsymbol{S}_1 - \boldsymbol{S}_3]\cdot[\boldsymbol{S}_2 - \boldsymbol{S}_4] \tag{5.15}$$

where the $\boldsymbol{S}_i$ $(i = 1,4)$ are the four spins surrounding the plaquet at position $\boldsymbol{R}$. In our mean field theory, we can estimate the Ising transition temperature by evaluating the temperature at which the Bose pairing becomes anisotropic. The general form of the Bose pairing field is

$$\Delta_{\boldsymbol{q}} = 2\tilde{\Delta}_1 c_y + 4\tilde{\Delta}_2 c_x c_y. \tag{5.16}$$

The development of a finite $\tilde{\Delta}_1$ results in the breaking of lattice symmetry. Extracting the coefficient of c_x in the the pairing equation (3.17) for $\Delta_{\boldsymbol{q}}$ we find

$$\tilde{\Delta}_1 = J_1 \int_{\boldsymbol{q}} \frac{c_y\tilde{\Delta}_{\boldsymbol{q}}}{2\omega_{\boldsymbol{q}}} \coth[\tfrac{1}{2}\beta\omega_{\boldsymbol{q}}]. \tag{5.17}$$

In the limit that $\tilde{\Delta}_1 \to 0$, we obtain an equation for the mean field Ising transition temperature $T_i = (\beta_i)^{-1}$

$$\frac{1}{J_1} = \int_{\boldsymbol{q}} \left\{ \frac{c_y^2}{\omega_{\boldsymbol{q}}} \coth(\tfrac{1}{2}\beta_i\omega_{\boldsymbol{q}}) + 2\tilde{\Delta}_2 c_x c_y^2 \frac{\partial}{\partial\tilde{\Delta}_1}\left(\frac{\coth(\frac{1}{2}\beta_i\omega_{\boldsymbol{q}})}{\omega_{\boldsymbol{q}}}\right)\right\} \tag{5.18}$$

where the differential is to be evaluated at $\tilde{\Delta}_1 = 0$. This equation provides a more quantitative estimate of the Ising transition temperature, than that provided by the qualitative condition $l_o(T_i) = \xi(T_i)$ used in the scaling theory treatment of this problem. As the gap Δ_o decreases, the last term in this expression diverges, so there is always an Ising transition, so long as the magnetic moment is finite. For small J_1, T_i becomes small, and the integral is dominated by the second term. Evaluating the logarithmic contributions, we find

$$\frac{1}{J_1} = \frac{4T_i}{\pi\Delta_o^2} \tag{5.19}$$

or

$$T_i = \frac{4\pi J_2 S^2}{\ln(\pi T_i / J_1 S^2)}. \tag{5.20}$$

A similar equation was previously derived from the scaling analysis. As J_1 is increased beyond this small critical value, the Ising temperature increases, rising to a maximum value near the fully frustrated point $J_1 = 2J_2$.

The spin-binding effects of 'order from disorder' on more general incommensurate magnetic structures are more striking. The leading order corrections to the Schwinger boson spectrum now lead to the development of *four* quantum exchange gaps at $\boldsymbol{q} = \boldsymbol{Q}$ (Δ_2) and at $\boldsymbol{q} = \boldsymbol{Q}'$ (Δ_1). Δ_1 is the quantum exchange gap that appears once the lattice symmetry of the environment of the spin condensate is broken, and the 'double twist' phason acquires a stiffness. The second gap Δ_2 appears at points $\boldsymbol{q} = \pm\boldsymbol{Q}$ where we would normally expect a Goldstone mode associated with the spin rotational invariance about an axes in the plane of the spins. As we shall see in detail in the next section, the disappearance of the Goldstone mode from the single spin wave spectrum is a signal of spin -wave *binding*. At energies below $\sim \Delta_2$, spin waves are bound into pairs with momentum $\pm\boldsymbol{Q}$, forming the twisted pair condensate. The spectral weight for the Goldstone mode at $\boldsymbol{q} = \pm\boldsymbol{Q}$ is accordingly transferred into the two-spin wave channel. The corresponding length scale $\xi_{\boldsymbol{Q}} \sim c/\Delta_2$ can be then loosely interpreted as a spin coherence length characterising the size of the bound-spin pairs. The binding gap Δ_2 can be interpreted as a pairing energy, or an excitation gap for out-of-phase rotations of the magnetisation and the twist pairing field. These gaps are given by

$$\Delta_i^2 = 2\tilde{h}^{(o)}_{\boldsymbol{Q}_i} \sum \{ (\mathcal{J}^+_{\boldsymbol{Q}_i \boldsymbol{q}} - \mathcal{J}^+_{0\boldsymbol{q}})\alpha^{(o)}_{\boldsymbol{q}} + (\mathcal{J}^-_{\boldsymbol{Q}_i \boldsymbol{q}} + \mathcal{J}^-_{0\boldsymbol{q}})\eta^{(o)}_{\boldsymbol{q}} \}. \tag{5.21}$$

Similar expressions can also be derived from leading order Spin Wave interactions. In spin wave theory, the recovery of the Goldstone mode at $\boldsymbol{q} = \pm Q$ involves consideration of cubic spin wave interactions that also lead to spin wave binding [32, 33]. In the next two sections, we shall see how we can deduce the properties of these long-wavelength modes by considering long-wavelength distortions of the ground state.

6. Goldstone mode structure: twist waves

The biaxial character of quantum helimagnets has interesting consequences for Goldstone modes which correspond to long-wavelength twists of the magnetisation. Classically, these distortions must be transverse to the local magnetisation, and local rotations about the magnetisation axes do not change the wavefunction of a helimagnet in the large-S limit. In a quantum helimagnet, local rotations about the magnetisation axis rotate the twist axis of the paired 'normal fluid', leading to an additional set of Goldstone modes. We can construct these modes by considering the action of long wavelength twists on the ground state. Uniform rotations about the x or y axis in the untwisted reference frame are non-uniform in the twisted reference frame

$$\begin{aligned} \sigma^z|\tilde{\Psi}\rangle &= P_S g^+ \sigma^z |\Psi\rangle \\ \sigma^\pm|\Psi\rangle &= P_S g^+ \sigma^\pm(\pm \boldsymbol{Q})|\Psi\rangle. \end{aligned} \tag{6.1}$$

These operations generate new states that are orthogonal, but degenerate with the ground state. Since the ground state is axially symmetric about the twist axis, the states $\sigma^-(\boldsymbol{Q})|\Psi\rangle$ and $\sigma^+(\boldsymbol{Q})|\Psi\rangle$ are also orthogonal to the original ground state.

We now construct the Goldstone modes of the ground-state by applying an infinitesimal rotation about the z axis in the twist reference frame, or infinitesimal twists $\sigma_x(\pm\boldsymbol{Q})$ and $\sigma_y(\pm\boldsymbol{Q})$ about the x and y axis. We define

$$\begin{aligned} \sigma^z|\Psi\rangle &= \sqrt{2}NS^* A^\dagger{}_3|\Psi\rangle \\ \mathrm{i}\sigma^y(\pm\boldsymbol{Q})|\Psi\rangle &= \sqrt{2}NS^* A^\dagger{}_2(\mp\boldsymbol{Q})|\Psi\rangle \\ \sigma^x(\pm\boldsymbol{Q})|\Psi\rangle &= \sqrt{2}NS^* A^\dagger{}_1(\mp\boldsymbol{Q})|\Psi\rangle \end{aligned} \tag{6.2}$$

where the prefactor $\sqrt{2}NS^*$ ensures a finite normalisation in the thermodynamic limit; then

$$\begin{aligned} A^\dagger{}_3 &= \frac{1}{\sqrt{2}}(a^\dagger{}_{o\uparrow} - a^\dagger{}_{o\downarrow}) \\ A^\dagger{}_2(\pm\boldsymbol{Q}) &= \sqrt{Z_{\boldsymbol{Q}}}\left[\frac{(a^\dagger{}_{\pm\boldsymbol{Q}\uparrow} - a^\dagger{}_{\pm\boldsymbol{Q}\downarrow})}{\sqrt{2}}\right] + \frac{1}{\sqrt{2NS^*}}\sum_{q\neq 0\ \sigma} v_{\boldsymbol{q}} u_{-\boldsymbol{q}\pm\boldsymbol{Q}}(a^\dagger{}_{\boldsymbol{q}\sigma}\sigma a^\dagger{}_{-\boldsymbol{q}\pm\boldsymbol{Q}\sigma}) \\ A^\dagger{}_1(\pm\boldsymbol{Q}) &= \sqrt{Z_{\boldsymbol{Q}}}\left[\frac{(a^\dagger{}_{\pm\boldsymbol{Q}\uparrow} + a^\dagger{}_{\pm\boldsymbol{Q}\downarrow})}{\sqrt{2}}\right] + \frac{1}{\sqrt{2NS^*}}\sum_{q\neq 0\ \sigma} v_{\boldsymbol{q}} u_{-\boldsymbol{q}\pm\boldsymbol{Q}}(a^\dagger{}_{\boldsymbol{q}\sigma} a^\dagger{}_{-\boldsymbol{q}\pm\boldsymbol{Q}\,\sigma}). \end{aligned} \tag{6.3}$$

Here we have rewritten the Bose fields $b^\dagger{}_{q\sigma}$ in terms of the spin quasiparticles and employed the result $a_{q\sigma}|\tilde{\Psi}\rangle = 0$. In the first expression, we have substituted $u_0 = \sqrt{NS^*}$. The last two transformations define the independent components of a rotations about axes perpendicular to the plane of the twist. The quantity $Z_{\boldsymbol{Q}}$ is a wavefunction renormalisation constant for the Goldstone modes, defining the overlap of the zero-mode for rotation about the $\hat{y}$ axis with the corresponding non-interacting spin wave:

$$Z_{\boldsymbol{Q}} = \langle\Psi|a_{\boldsymbol{Q}}|\boldsymbol{Q};y\rangle \tag{6.4}$$

where $a_{\boldsymbol{Q}} = (a_{\boldsymbol{Q}\uparrow} - a_{\boldsymbol{Q}\downarrow})/\sqrt{2}$ and $|\boldsymbol{Q};y\rangle = A^\dagger{}_2|\tilde{\Psi}\rangle$. For the classical helimagnet, this quantity is unity, but at finite S, due to the spin wave pairing, $u_{\boldsymbol{Q}}$ is finite and the spectral weight of the Goldstone mode in the one-magnon channel vanishes in the thermodynamic limit

$$Z_{\boldsymbol{Q}} = \frac{u_{\boldsymbol{Q}}^2}{NS^*} \to 0. \tag{6.5}$$

We conclude that the spectral weight of the Goldstone mode is entirely transferred to the the two-magnon channel; and equation (6.3) becomes

$$\begin{aligned} A^\dagger{}_3 &= \frac{1}{\sqrt{2}}(a^\dagger{}_{o\uparrow} - a^\dagger{}_{o\downarrow}) \\ A^\dagger{}_2(\pm \boldsymbol{Q}) &= \frac{1}{\sqrt{2}NS^*}\sum_{\boldsymbol{q}\,\sigma} v_{\boldsymbol{q}} u_{-\boldsymbol{q}\pm\boldsymbol{Q}}(a^\dagger{}_{\boldsymbol{q}\,\sigma}\sigma a^\dagger{}_{-\boldsymbol{q}\pm\boldsymbol{Q}\,\sigma}) \\ A^\dagger{}_1(\pm \boldsymbol{Q}) &= \frac{1}{\sqrt{2}NS^*}\sum_{\boldsymbol{q}\,\sigma} v_{\boldsymbol{q}} u_{-\boldsymbol{q}\pm\boldsymbol{Q}}(a^\dagger{}_{\boldsymbol{q}\,\sigma} a^\dagger{}_{-\boldsymbol{q}\pm\boldsymbol{Q}\,\sigma}). \end{aligned} \tag{6.6}$$

Equations (6.6) are the generalisations of the Goldstone modes that appear in spin wave theory. We note that $A_1{}^\dagger(\pm\boldsymbol{Q})$ corresponds to a twist of the normal spin fluid about the magnetisation axis, and is *absent* in the uniaxial classical magnet.

Physical spin excitations above the fully projected wavefunction correspond to the creation of particle-hole *pairs* of the unconstrained excitations considered above, one of which is absorbed by the vacuum. If we were to project out the singlet component of the original ground-state, then the action of the total spin operator on the ground-state produces an excitation of spin 1. We can interpret the excitation $A^\dagger{}_3$ as a single delocalised spin flip. The excitations $A^\dagger{}_1$ and $A^\dagger{}_2$ are 'double-spin flip' excitations, corresponding to a bound pair of magnons with an antisymmetric wavefunction and total spin $S = 1$. The double spin-flip nature of the Goldstone modes is associated with the formation of 'pseudo-vector' order associated with the twist in the normal fluid of spins. These modes do not require the presence of a sublattice magnetisation, and might be called 'twistons'. An alternative way to think about these $S = 1$ twiston excitations is to regard them as collisionless versions of a second sound-spin wave [11]. In a three dimensional helimagnet, the effect of temperature will be rather similar to that of quantum fluctuations, and the hydrodynamic version of these modes will constitute a second-spin wave.

In the next section, we discuss the appearance of these three long-wavelength modes in the long-wavelength action for a quantum helimagnet. Lastly, note that in the limit where the sublattice magnetisation S^* vanishes, the first mode ($A^\dagger{}_3$) disappears, whilst the other two modes will become degenerate. These are then the Goldstone modes of a spin nematic, to be discussed in the last section.

7. Long-wavelength behaviour: gauge modes and sigma models

In this section we focus on the long-wavelength behaviour of spin systems, which we treat by direct analogy with the Landau–Ginzberg approach to long-wavelength modes in a superfluid. Rather than computing the spinwave stiffness by explicitly distorting the orientations of the spins, we appeal to the rotational gauge invariance and introduce a spin vector potential field that is gauge equivalent to a twist of the spins. The spin wave stiffness is then the susceptibility that relates an external spin-vector potential to an induced spin current. This permits us to treat the spin current and the magnetisation on an equal footing. We shall show how it is possible to relate the microscopic motions of the spin quanta to the macroscopic susceptibilities. In particular, we shall show how the biaxial nature of helimagnets, and the uniaxial behaviour of collinear magnets and spin nematics enters in our approach.

Let χ be the susceptibility tensor, and γ^l the spin wave stiffness tensors for each direction ($l = 1, d$), then the long-wavelength magnetic response can be written

$$\begin{aligned} \boldsymbol{M} &= -\partial F/\partial \boldsymbol{B} = \boldsymbol{\chi}\cdot\boldsymbol{B} \\ \mathcal{J}_l &= -\partial F/\partial \boldsymbol{A}_l = -\gamma^l\cdot\boldsymbol{A}_l \end{aligned} \tag{7.1}$$

where $\boldsymbol{M}$ is the magnetisation and $\mathcal{J}_l$ is the spin current in direction l. The spin wave stiffness tensor γ^l is then the spin analogue of the London Kernel that relates vector potential to mass current in superfluids. For compactness we shall use a four vector notation

$$\begin{aligned} \mathcal{J}_\mu &= (\boldsymbol{M}, \boldsymbol{j}_l) \\ \gamma^\mu &= (-\boldsymbol{\chi}, \gamma^l) \qquad (l = 1, d). \end{aligned} \tag{7.2}$$

We begin by considering the rotationally covariant form of the action for a two-dimensional antiferromagnet. In non-covariant form, the action is a non-linear sigma model

$$I = \frac{JS^2}{2}\int \mathrm{d}^d x\,\mathrm{d}t\left\{-\frac{1}{c^2}(\partial_t\hat{\boldsymbol{n}})^2 + (\nabla\hat{\boldsymbol{n}})^2\right\}. \tag{7.3}$$

Now suppose we rotate to a rotated reference frame, defined by $\hat{\boldsymbol{n}} \to \exp(-\boldsymbol{\theta}\times)\hat{\boldsymbol{n}}$, and introduce gauge fields $(\boldsymbol{B}, \boldsymbol{A}_l) \sim -\exp(-\boldsymbol{\theta}\times)(\partial_t, \nabla_l)\boldsymbol{\theta}$, ($l = 1, d$) then the gauge covariant form of the action is

$$\begin{aligned} I &= \frac{JS^2}{2}\int \mathrm{d}^d x\,\mathrm{d}t\left\{-\frac{1}{c^2}(\partial\!\!\!/_t\hat{\boldsymbol{n}})^2 + (\nabla\!\!\!\!/\,\hat{\boldsymbol{n}})^2\right\} \\ \partial\!\!\!/_t &= (\partial_t + \boldsymbol{B}\times) \qquad \nabla\!\!\!\!/_l = (\nabla_l + \boldsymbol{A}_l\times). \end{aligned} \tag{7.4}$$

In this form, the action has the gauge invariance

$$\begin{aligned} \hat{\boldsymbol{n}} &\to \hat{\boldsymbol{n}} - \boldsymbol{\theta}\times\hat{\boldsymbol{n}} \\ \boldsymbol{B} &\to \boldsymbol{B} + \partial\!\!\!/_t \\ \boldsymbol{A}_l &\to \boldsymbol{A}_l + \nabla\!\!\!\!/_l. \end{aligned} \tag{7.5}$$

Classically, the spin current and the magnetisation are then given by

$$\begin{aligned} \boldsymbol{M} &= -\partial S/\partial \boldsymbol{B} = \chi_\perp[\hat{\boldsymbol{n}}\times\partial_t\hat{\boldsymbol{n}} + \boldsymbol{B}_\perp] \\ \mathcal{J}_l &= -\partial S/\partial \boldsymbol{A}_l = -\gamma_\perp[\hat{\boldsymbol{n}}\times\nabla_l\hat{\boldsymbol{n}} + (\boldsymbol{A}_l)_\perp] \end{aligned} \tag{7.6}$$

where

$$\chi_\perp = JS^2/c^2 \qquad \gamma_\perp = JS^2 \tag{7.7}$$

illustrating that the spin wave stiffness is space and time are the susceptibilities of the magnetic field and the spin vector potential.

We now generalise these considerations to a helimagnet. Locally, the magnetisation of a helimagnet precesses in space

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{\imath}} \cos \boldsymbol{Q} \cdot \boldsymbol{R} + \hat{\boldsymbol{\jmath}} \sin \boldsymbol{Q} \cdot \boldsymbol{R} \tag{7.8}$$

where $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}})$ form a local Cartesian co-ordinate system. The magnetisation axis $\hat{\boldsymbol{n}}$ and the twist axis $\hat{\boldsymbol{k}}$ then define a biaxial SO(3) order parameter. It is useful to define the principal axes

$$(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3) = (\hat{\boldsymbol{n}}, \hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}}, \hat{\boldsymbol{k}}). \tag{7.9}$$

In the ground state, these vectors precess in space according to $\nabla_l \hat{\boldsymbol{e}}_\lambda = Q_l \hat{\boldsymbol{e}}_3 \times \hat{\boldsymbol{e}}_\lambda$ where the Q_l are the components of the incommensurate magnetic wavevector. The covariant derivative must now be modified to accommodate the spontaneous presence of a twist $\boldsymbol{A}_l \rightarrow \boldsymbol{A}_l - Q_l \hat{\boldsymbol{e}}_3$, so that

$$\not{\nabla}_l = \nabla_l + \boldsymbol{A}_l - Q_l \hat{\boldsymbol{e}}_3. \tag{7.10}$$

In the long-wavelength limit, the action is expanded to Gaussian order in the deviation of the precession rate from its equilibrium value. Writing

$$\boldsymbol{\omega}_\mu \times \hat{\boldsymbol{e}}_\lambda = (\not{\partial}_t, \not{\nabla}_l) \hat{\boldsymbol{e}}_\lambda \qquad (\mu = 0, d) \tag{7.11}$$

then the generalisation of the long-wavelength action to a helimagnet is

$$I = \frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}t \, \{ -\chi_\lambda (\omega_o^\lambda)^2 + \gamma_\lambda^l (\omega_l^\lambda)^2 \} \tag{7.12}$$

where the space-time precession vectors are resolved along the local principal axes $\boldsymbol{\omega}_\mu = \omega_\mu^\lambda \hat{\boldsymbol{e}}^\lambda$. It is useful to divide the susceptibilities into a contribution associated with the spin condensate and a part associated with the twisted normal spin fluid

$$\gamma_\lambda^\mu = [\gamma_\lambda^\mu]_\mathrm{c} + [\gamma_\lambda^\mu]_\mathrm{n}. \tag{7.13}$$

The spin condensate behaves as classical fixed length spins, so the susceptibilities parallel to the magnetisation axis vanish

$$[\gamma_1^\mu]_\mathrm{c} = 0. \tag{7.14}$$

Since the normal spin fluid is uniaxial, the susceptibilities perpendicular to the twist are equal, whilst the normal state stiffness along the twist axis is zero

$$[\gamma_1^\mu]_\mathrm{n} = [\gamma_2^\mu]_\mathrm{n} \qquad [\gamma_3^l]_\mathrm{n} = 0. \tag{7.15}$$

In the large-S limit, the normal component of the generalised susceptibilities vanishes, so the longitudinal susceptibilities are zero; the ratio of the transverse spin wave stiffness to the transverse magnetic suceptibilities determines the classical spin wave velocities

$$(c_2^{(l)})^2 = \gamma_2^l / \chi_2 \qquad (c_3^{(l)})^2 = \gamma_3^l / \chi_3. \tag{7.16}$$

In a helimagnetic structure at finite S, the normal component of the stiffness is finite and the ratio of the corresponding suceptibilities determines the velocity of the 'twist wave' c_1 associated with twisting the normal fluid about the magnetisation axis

$$(c_1^{(l)})^2 = \gamma_1^l/\chi_1. \tag{7.17}$$

The magnetic susceptibility and spin wave stiffness tensors are determined by the response to the gauge field $A_\mu = (\boldsymbol{B}, \boldsymbol{A}_l)$

$$[\gamma^\mu]_{\alpha\beta} = \frac{\partial^2 I}{\partial A^\alpha_\mu \partial A^\beta_\mu} = \sum_{\lambda=1,3} \gamma^\mu_\lambda [\hat{e}_\lambda]^\alpha [\hat{e}_\lambda]^\beta. \tag{7.18}$$

Unlike the antiferromagnet, however, the principal axes precess in space, giving rise to a non-uniform component in the magnetic susceptibility and stiffness tensors. In the basis $(\hat{\mathrm{i}}, \hat{\mathrm{j}}, \hat{\mathrm{k}})$, the susceptibility tensor has the form

$$\gamma^\mu = \gamma^\mu_U + \gamma_{2Q}(\boldsymbol{x}) \qquad \gamma^\mu_U = \begin{bmatrix} \gamma^\mu_{(+)} & & \\ & \gamma^\mu_{(+)} & \\ & & \gamma^\mu_3 \end{bmatrix} \qquad \gamma^\mu_{2Q}(\boldsymbol{x}) = \gamma^\mu_{(-)} \begin{bmatrix} \tilde{c} & \tilde{s} & \\ \tilde{s} & -\tilde{c} & \\ & & 0 \end{bmatrix} \tag{7.19}$$

where

$$\gamma^\mu_{(\pm)} = \tfrac{1}{2}(\gamma^\mu_1 \pm \gamma^\mu_2) \tag{7.20}$$

and $(\tilde{c}, \tilde{s}) = (\cos 2\boldsymbol{Q}\cdot\boldsymbol{x}, \sin 2\boldsymbol{Q}\cdot\boldsymbol{x})$. In a uniform external magnetic field $\boldsymbol{B}$, a *non-uniform* magnetisation develops with wavevector $2\boldsymbol{Q}$: $\boldsymbol{M}_{2Q} = \chi_{2Q}(\boldsymbol{x})\cdot\boldsymbol{B}$ [34]. The rotating component of the susceptibility is transferred to the magnetic permittivity of the system: $\mu(\boldsymbol{r}) = \mu + 4\pi\boldsymbol{\chi}(\boldsymbol{r})$, which gives rise to optical activity, as discussed in the final section.

The non-uniform response is an appropriate order parameter for biaxial magnetic behaviour. There are actually two special uniaxial limits of the above behaviour where the non-uniform response vanishes: collinear antiferromagnets and spin nematics [15]. The special limit of the collinear magnet occurs when $2\boldsymbol{Q} \boxminus 0$ and the twist vanishes. In this case, the normal component of the spinwave stiffness vanishes $[\gamma^l_l]_{\mathrm{n}} = 0$, and the magnetic susceptibilities transverse to the magnetisation are equal $\chi_{2,3} = \chi_\perp$. In spin nematics, as discussed in detail in the next section, the magnetisation vanishes, but the spin fluid remains twisted. The classical component of the stiffness and magnetic susceptibility accordingly vanish and the stiffness along the twist axis (γ_3) is zero.

Let us now consider how these various terms appear from our microscopic calculation. If we expand the gauge invariant form of the Hamiltonian

$$H[\boldsymbol{A}_l] = \frac{1}{2}\sum J_{ij}\boldsymbol{S}_i \exp\left(-\int_j^i \boldsymbol{A}_l \, \mathrm{d}R_l \times\right)\boldsymbol{S}_j - \sum_j \boldsymbol{B}_j\cdot\boldsymbol{S}_j \tag{7.21}$$

in powers of the spin vector potential, we may write

$$H[\boldsymbol{A}_l] = H - \sum_{\boldsymbol{x}} \boldsymbol{A}_l(\boldsymbol{x})\boldsymbol{j}_l(\boldsymbol{x}) + \frac{1}{2}\sum_{\boldsymbol{x}} A^\alpha_l(\boldsymbol{x})\mathcal{N}^l_{\alpha\beta}(\boldsymbol{x})A^\beta_l(\boldsymbol{x}). \tag{7.22}$$

Here

$$j_l(\boldsymbol{x}) = \frac{1}{2}\sum_{\boldsymbol{R}} J(\boldsymbol{R})R_l \boldsymbol{S}(\boldsymbol{x}) \times \boldsymbol{S}(\boldsymbol{x}+\boldsymbol{R})$$
$$\mathcal{N}^l_{\alpha\beta}(\boldsymbol{x}) = \frac{1}{2}\sum_{\boldsymbol{R}} J(\boldsymbol{R})(R_l)^2[S^\alpha(\boldsymbol{x})S^\beta(\boldsymbol{x}+\boldsymbol{R}) - \delta^{\alpha\beta}\boldsymbol{S}(\boldsymbol{x})\cdot\boldsymbol{S}(\boldsymbol{x}+\boldsymbol{R})]. \tag{7.23}$$

The derivative with respect to the spin vector potential yields the spin current

$$\mathcal{J}_l = j_l - \mathcal{N}^l \cdot \boldsymbol{A}_l \qquad ([\mathcal{N}^l]_{\alpha\beta} = \mathcal{N}^l_{\alpha\beta}). \tag{7.24}$$

By analogy with superconductors, the second term is the 'diamagnetic' part of the spin current, which gives rise to an instantaneous response to an external twist field. The second term is the 'paramagnetic' spin current which develops as the spin wavefunction responds adiabatically to the presence of the external twist field. From the second derivative of the partition function we may compute the linear response to a spin vector potential in terms of a diamagnetic and a paramagnetic contribution

$$\mathcal{J}_l(x) = -\sum_{\boldsymbol{x}'}\int_{\tau'} \gamma^l(\boldsymbol{x},\boldsymbol{x}';\tau-\tau')\boldsymbol{A}_l(x')$$
$$\gamma^l(\boldsymbol{x},\boldsymbol{x}';\tau-\tau') = \mathcal{N}^l(\boldsymbol{x}')\delta^3(x-x') - \langle \mathrm{T} j_l(x) j_l(x')\rangle \tag{7.25}$$

where we have used the shorthand $x \boxminus (\boldsymbol{x},\tau)$, $\delta^3(x) \equiv \delta_{\boldsymbol{x}}\delta(\tau)$ and suppressed the spin indices. In a spin fluid with unbroken rotational invariance, the long time paramagnetic response must exactly cancel the diamagnetic term at long times and long-distances, renormalising the spin wave stiffness to zero. When rotational gauge invariance is broken, this cancellation is no longer perfect, leading to certain non-vanishing components of the stiffness tensor. This behaviour is completely analogous to neutral superfluids. Note that the spin analogue of charged superfluids does not exist, since there are no dynamical spin gauge fields. If twist gauge field was dynamical, then this incomplete cancellation would lead to the spin analogue of the Meissner effect.

In a classical helimagnet $\boldsymbol{S}(x) = S\hat{\boldsymbol{e}}_1(x)$ (7.8), and

$$\langle S(\boldsymbol{x})^\alpha S(\boldsymbol{x}+\boldsymbol{R})^\beta\rangle = S^2[\hat{e}_1(\boldsymbol{x})]^\alpha\{[\hat{e}_1(\boldsymbol{x})]^\beta \cos\boldsymbol{Q}\cdot\boldsymbol{R} + [\hat{e}_2(\boldsymbol{x})]^\beta \sin\boldsymbol{Q}\cdot\boldsymbol{R}\}. \tag{7.26}$$

Substituting into the expressions for the spin current, and 'diamagnetic' stiffness we find

$$\langle j_l\rangle = \frac{S^2}{2}\sum_{\boldsymbol{R}} R_l \sin\boldsymbol{Q}\cdot\boldsymbol{R} = -\frac{S^2}{2}\nabla_{\boldsymbol{Q}} J(\boldsymbol{Q}) = 0$$
$$\mathcal{N}^l_{\alpha\beta}(x) = S^2 \sum_\lambda \tilde{\mathcal{N}}^l_\lambda [\hat{e}_\lambda(x)]^\alpha [\hat{e}_\lambda(x)]^\beta \tag{7.27}$$

where

$$\tilde{\mathcal{N}}^l_a = \begin{cases} 0 & (a=1) \\ \frac{1}{2}\nabla_l^2 J(\boldsymbol{Q}) & (a=2,3). \end{cases} \tag{7.28}$$

The first equation in (7.27) indicates that although there are local spin currents along individual bonds inside the helimagnet, the long-wavelength uniform component to these currents is absent in equilibrium, as expected on physical grounds. The absence of a classical stiffness about the magnetisation axis ($\tilde{\mathcal{N}}_1^l = 0$) is evidence of the uniaxial nature of the classical helimagnet. Finally, note that there are no 'diamagnetic' contributions to the susceptibility ($\mathcal{N}_\mu = (0, \mathcal{N}_l)$), as the magnetic field couples linearly to the spins.

Once quantum fluctuations are reintroduced, the paramagnetic part of the stiffness develops and the anisotropic short-range spin fluctuations impart biaxial character to the stiffness tensor. We now compute these corrections in the Schwinger boson scheme. We shall divide the diamagnetic component of the stiffness $\mathcal{N}^l$ into a condensate component and a component associated with the anisotropic nature of the 'normal' fluid

$$\mathcal{N}^l(\boldsymbol{x}) = N^l + [S^*]^2 \tilde{\mathcal{N}}^l(\boldsymbol{x}) \tag{7.29}$$

where S^* is the renormalised magnetisation and the condensate part has the form given in (7.27). To calculate N^l and the fluctuating 'paramagnetic' component to the susceptibilities, we need to express the paramagnetic component of the spin current in terms of the Bose fields. For this purpose, we adopt a Ballian Werthammer [14] notation for the Schwinger boson fields, writing

$$\Psi_{\boldsymbol{q}} = \begin{pmatrix} b_{\boldsymbol{q}} \\ \sigma_2 b^\dagger{}_{-\boldsymbol{q}} \end{pmatrix} = \begin{pmatrix} b_{\boldsymbol{q}\uparrow} \\ b_{\boldsymbol{q}\downarrow} \\ -\mathrm{i} b^\dagger{}_{-\boldsymbol{q}\downarrow} \\ \mathrm{i} b^\dagger{}_{-\boldsymbol{q}\uparrow} \end{pmatrix}. \tag{7.30}$$

The commutation algebra of these spinors is

$$[\Psi_{\boldsymbol{q}}, \Psi^\dagger{}'_{\boldsymbol{q}}]_{\alpha\beta} = \delta_{\boldsymbol{q}\boldsymbol{q}'}[\tau_3 \otimes \mathbf{1}]_{\alpha\beta} \tag{7.31}$$

where the matrices $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ denote Pauli matrices that act in particle-hole space. In this form, the mean field Hamiltonian can be written

$$\begin{aligned} \hat{H} &= \sum_{\boldsymbol{q} \in \frac{1}{2}\mathrm{BZ}} \Psi^\dagger{}_{\boldsymbol{q}} \tau_3 \mathcal{H}(\boldsymbol{q}) \Psi_{\boldsymbol{q}} \\ \mathcal{H}(\boldsymbol{q}) &= [\tilde{h}(\boldsymbol{q})\, \tau_3 - \mathrm{i}\Delta(\boldsymbol{q})\, \tau_1 \otimes \hat{\boldsymbol{k}} \cdot \boldsymbol{\sigma}] \end{aligned} \tag{7.32}$$

where the $\boldsymbol{q}$ vector associated with the Bose fields is restricted to on half the Brillouin zone to avoid double counting. The mean field propagator is then

$$\begin{aligned} \langle T\Psi_{\boldsymbol{q}}(\tau)\Psi^\dagger{}_{\boldsymbol{q}}(0)\rangle &= -T\sum_{\mathrm{n}} \mathcal{G}(\boldsymbol{q}, \mathrm{i}\nu_{\mathrm{n}})\tau_3 \exp(-\mathrm{i}\nu_{\mathrm{n}}\tau) \\ \mathcal{G}(\boldsymbol{q}, \mathrm{i}\nu_{\mathrm{n}}) &= [\mathrm{i}\nu_{\mathrm{n}} - \mathcal{H}(\boldsymbol{q})]^{-1} = \sum_{\alpha=\pm 1} \frac{1}{\mathrm{i}\nu_{\mathrm{n}} - \alpha\omega_{\boldsymbol{q}}}\, \mathrm{P}_\alpha(\boldsymbol{q}) \end{aligned} \tag{7.33}$$

where the projection operator is

$$\mathrm{P}_\pm(\boldsymbol{q}) = \tfrac{1}{2} \pm \tfrac{1}{2}[(u_{\boldsymbol{q}}^2 + v_{\boldsymbol{q}}^2)\tau_3 - 2\mathrm{i}u_{\boldsymbol{q}}v_{\boldsymbol{q}}\, \tau_1 \otimes \sigma_3)] \tag{7.34}$$

with $u_{\boldsymbol{q}}$ and $v_{\boldsymbol{q}}$ as defined in (4.1). A useful quantity to know is the density matrix, which can be obtained directly from the Green function

$$\begin{aligned}\langle \Psi_{\boldsymbol{q}}^{\dagger\beta} \tau_3 \Psi_{\boldsymbol{q}}^{\alpha} \rangle &= \left\{ n(\omega_{\boldsymbol{q}}) P_+(\boldsymbol{q}) - (n(\omega_{\boldsymbol{q}})+1) P_-(\boldsymbol{q}) \right\}^{\alpha\beta} \\ &= \tfrac{1}{2}\left\{ 1 + \coth[\tfrac{1}{2}\beta\omega_{\boldsymbol{q}}][(u_{\boldsymbol{q}}^2 + v_{\boldsymbol{q}}^2)\tau_3 - 2i u_{\boldsymbol{q}} v_{\boldsymbol{q}} \tau_1 \otimes \sigma_3)] \right\}^{\alpha\beta}.\end{aligned} \tag{7.35}$$

The relationship between the Bose operators in the twisted (b) and untwisted ($\tilde{b}$) spin reference frame is

$$\tilde{\Psi}^{\dagger}{}_{\boldsymbol{k}} = \sum_{\sigma} \Psi^{\dagger}{}_{\boldsymbol{k}+\sigma \boldsymbol{Q}/2} \mathcal{P}_{\sigma} \qquad (\sigma = \pm 1) \tag{7.36}$$

where $\mathcal{P}_{\sigma} = [1 + \sigma\sigma^3]/2$ projects out the up and down spin components of the spinor, so in the untwisted reference frame, the Hamiltonian $\mathcal{H}(\boldsymbol{q})$ is replaced by $\mathcal{H}(\boldsymbol{q}+\sigma\boldsymbol{Q}/2)$. Suppose we introduce an external magnetic field and an external twist field $\boldsymbol{A}_l$, then $q_l \to q_l - \boldsymbol{\sigma}\cdot\boldsymbol{A}_l/2$, and the spin fluid Hamiltonian becomes

$$\hat{H}'[A] = \sum_{\boldsymbol{q}\in\frac{1}{2}\mathrm{BZ}} \tilde{\Psi}^{\dagger}{}_{\boldsymbol{q}} \tau_3 \mathcal{H}[q_l + (Q_l \sigma^3 - \boldsymbol{A}_l \cdot \boldsymbol{\sigma})/2] \tilde{\Psi}_{\boldsymbol{q}} - \sum_{\boldsymbol{q}} \boldsymbol{B}_{\boldsymbol{q}} \cdot \boldsymbol{S}_{-\boldsymbol{q}} \tag{7.37}$$

where

$$\boldsymbol{S}_{\boldsymbol{q}} = \sum_{\boldsymbol{k}\in\frac{1}{2}\mathrm{BZ}} \tilde{\Psi}^{\dagger}{}_{\boldsymbol{k}-\boldsymbol{q}/2}\, \tau_3 \frac{\boldsymbol{\sigma}}{2} \tilde{\Psi}_{\boldsymbol{k}+\boldsymbol{q}/2}. \tag{7.38}$$

To determine the spin currents and the intrinsic stiffness of the paired spin fluid, we expand (7.37) in a gradient expansion to second order

$$\hat{H}[A] = \hat{H} - \sum_{\boldsymbol{R}} \mathcal{A}_{\mu}(\boldsymbol{R}) \cdot \boldsymbol{j}_{\mu}(\boldsymbol{R}) + \frac{1}{2} \sum_{\boldsymbol{R}} \boldsymbol{A}_l(\boldsymbol{R}) \cdot N^l \cdot \boldsymbol{A}_l(\boldsymbol{R}) + \mathrm{O}(A^3) \tag{7.39}$$

where $\mathcal{A}_{\mu} = (\boldsymbol{B}, \boldsymbol{A}_l)$ and $\boldsymbol{j}_{\mu}(\boldsymbol{R}) = (\boldsymbol{S}(\boldsymbol{R}), \boldsymbol{j}_l(\boldsymbol{R}))$, and

$$\begin{aligned}\boldsymbol{j}_l(\boldsymbol{q}) &= \sum_{\boldsymbol{k}\in\frac{1}{2}\mathrm{BZ}} \tilde{\Psi}^{\dagger}{}_{\boldsymbol{k}-\boldsymbol{q}/2}\, \tau_3 \left\{ \frac{\boldsymbol{\sigma}}{4}, \nabla_l \mathcal{H}\left(q + \frac{\sigma_3}{2} Q\right) \right\} \tilde{\Psi}_{\boldsymbol{k}+\boldsymbol{q}/2} \\ N^l_{ab} &= \sum_{\boldsymbol{q}\in\frac{1}{2}\mathrm{BZ}} \left\langle \tilde{\Psi}^{\dagger}{}_{\boldsymbol{q}}\, \tau_3 \left[\frac{\sigma^a}{4}, \left\{ \frac{\sigma^b}{4}, \nabla_l^2 \mathcal{H}\left(q + \frac{\sigma_3}{2} Q\right) \right\} \right] \tilde{\Psi}_{\boldsymbol{k}} \right\rangle.\end{aligned} \tag{7.40}$$

Rewriting N^l in terms of the twisted Bose fields, and substituting expression (7.35) for the density matrix gives

$$\begin{aligned}N^l_{zz} &= \sum_{\boldsymbol{q}} \left[\frac{\tilde{h}_{\boldsymbol{q}} \nabla_l^2 h_{\boldsymbol{q}} - \Delta_{\boldsymbol{q}} \nabla_l^2 \Delta_{\boldsymbol{q}}}{4\omega_{\boldsymbol{q}}} \right] \\ N^l_{\perp} &= \sum_{\boldsymbol{q}} \left[\frac{\tilde{h}_{\boldsymbol{q}} \nabla_l^2 h^*_{\boldsymbol{q}} - \Delta_{\boldsymbol{q}} \nabla_l^2 \Delta^*_{\boldsymbol{q}}}{4\omega_{\boldsymbol{q}}} \right] \qquad (\perp \equiv xx, yy)\end{aligned} \tag{7.41}$$

where

$$h^*_{\boldsymbol{q}} = \tfrac{1}{4}(2h_{\boldsymbol{q}} + (h_{\boldsymbol{q}+\boldsymbol{Q}} + h_{\boldsymbol{q}-\boldsymbol{Q}}))$$
$$\Delta^*_{\boldsymbol{q}} = \tfrac{1}{4}(2\Delta_{\boldsymbol{q}} - (\Delta_{\boldsymbol{q}+\boldsymbol{Q}} + \Delta_{\boldsymbol{q}-\boldsymbol{Q}})). \tag{7.42}$$

Next, consider the spin currents. Resolving them along the rotating reference axes

$$\boldsymbol{j}_\mu(\boldsymbol{R}) = \sum_a j^a_\mu(\boldsymbol{R})\hat{e}_a(\boldsymbol{R}) \tag{7.43}$$

then the Fourier transform of $j^a_\mu(\boldsymbol{R})$ can be written (see appendix)

$$j^a_\mu(q) = \sum_{\boldsymbol{k}\in\ \frac{1}{2}\mathrm{BZ}} \Psi^\dagger{}_{\boldsymbol{k}-\boldsymbol{q}/2}\tau_3 J^a_\mu(q)\Psi_{\boldsymbol{k}+\boldsymbol{q}/2}. \tag{7.44}$$

Here the matrix elements are

$$\left(J^3_o(\boldsymbol{q}), J^3_l(\boldsymbol{q})\right) = \left(\tfrac{1}{2}\sigma^3, \tfrac{1}{2}\sigma^3\nabla_l\mathcal{H}(\boldsymbol{q})\right)$$
$$\left(J^\pm_o(\boldsymbol{q}), J^\pm_l(\boldsymbol{q})\right) = \left(\sigma^\pm, \sigma^\pm\nabla_l\tilde{\mathcal{H}}(\boldsymbol{q})\right) \tag{7.45}$$

and

$$\tilde{\mathcal{H}}(\boldsymbol{q}) = [h^+(\boldsymbol{q})\ \tau_3 + \mathrm{i}\Delta^-(\boldsymbol{q})\,\tau_1]$$
$$h^+_{\boldsymbol{q}} = \tfrac{1}{2}(h_{\boldsymbol{q}+\boldsymbol{Q}/2} + h_{\boldsymbol{q}-\boldsymbol{Q}/2}) \tag{7.46}$$
$$\Delta^-_{\boldsymbol{q}} = \tfrac{1}{2}(\Delta_{\boldsymbol{q}+\boldsymbol{Q}/2} - \Delta_{\boldsymbol{q}-\boldsymbol{Q}/2})$$

and $j^\pm_\mu = j^1_\mu \pm \mathrm{i}j^2_\mu$, $\sigma^\pm = \frac{1}{2}[\sigma^1 \pm \mathrm{i}\sigma^2]$.

In the twisted reference frame, the dynamical spin correlations are uniform and diagonal

$$\langle \mathrm{T}j_\mu{}^\lambda\ (R)j_\mu{}^\lambda\ (0)\rangle = T\sum_q \tilde{\gamma}^\mu_\lambda(q)\exp[\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{R} - \nu_\mathrm{n}\tau)] \tag{7.47}$$

where $R \equiv (\boldsymbol{R}, \tau)$, $q = (\boldsymbol{q}, \mathrm{i}\nu_\mathrm{n})$ and $\tilde{\gamma}^\mu_\lambda = \langle j'^\lambda_\mu\ (q)j'^\lambda_\mu(-q)\rangle$. In the untwisted reference frame, spin correlations take the form

$$\langle \mathrm{T}\boldsymbol{j}_\mu(x')\boldsymbol{j}_\mu(x)\rangle = \sum_\lambda \hat{e}_\lambda(x')\hat{e}_\lambda(x)\,\langle \mathrm{T}j^\lambda_\mu(x-x')j^\lambda_\mu(0)\rangle. \tag{7.48}$$

Averaging over $x - x' = R$ gives

$$\frac{1}{N\beta}\sum_{\boldsymbol{R}}\int \mathrm{d}\tau\,\langle \mathrm{T}j_\mu(x+R)j_\mu(x)\rangle = \tilde{\gamma}^\mu{}_\lambda\hat{e}_\lambda(x)\hat{e}_\lambda(x) \tag{7.49}$$

where spin indices have been suppressed, and

$$\tilde{\gamma}^\mu{}_\lambda = \begin{cases} \tilde{\gamma}^\mu{}_\lambda(\boldsymbol{Q}) & (\lambda = 1, 2) \\ \tilde{\gamma}^\mu{}_\lambda(0) & (\lambda = 3). \end{cases} \tag{7.50}$$

The full expression for the moments of the stiffness tensor is then

$$\gamma^{\mu}_{\lambda} = \mathcal{N}^{\mu}_{\lambda} - \tilde{\gamma}^{\mu}{}_{\lambda}. \tag{7.51}$$

The paramagnetic spin current correlations can be determined to one loop approximation from the Bose Green functions

$$\tilde{\gamma}^{\mu}_{a}(q) = T \sum_{\boldsymbol{k}\in\frac{1}{2}\mathrm{BZ}\,\mathrm{i}\Omega_n} \mathrm{Tr}[\mathcal{G}(k-q/2)J^{a}_{\mu}(k)\mathcal{G}(k+q/2)J^{a}_{\mu}(k)] \tag{7.52}$$

where $k = (\boldsymbol{k}, \mathrm{i}\Omega_{\mathrm{n}})$. Once a spin condensate develops, the Bose field acquires an expectation value, and the fluctuations may be separated into a condensate and a normal component. To compute the normal component to the fluctuations, it is sufficient to take the zero temperature limit of (7.52), neglecting the zero frequency poles in the Bose functions and setting $\coth[\beta\omega_{\boldsymbol{q}}/2]$ by one in the final result. Evaluating the Matsubara sums in (7.52), the zero temperature normal component of the zero frequency current correlation functions is then

$$\begin{aligned} \tilde{\gamma}^{\mu}_{ab}(\boldsymbol{q})_{\mathrm{n}} &= \sum_{\boldsymbol{k}} \frac{1}{(\omega_{\boldsymbol{k}^+} + \omega_{\boldsymbol{k}^-})} \Gamma^{\mu}_{ab}(\boldsymbol{k}^+, \boldsymbol{k}^-) \\ \Gamma^{\mu}_{ab}(\boldsymbol{k}^+, \boldsymbol{k}^-) &= \mathrm{Tr}[\mathrm{P}_+(\boldsymbol{k}^+)J^{a}_{\mu}(\boldsymbol{k})\mathrm{P}_-(\boldsymbol{k}^-)J^{b}_{\mu}(\boldsymbol{k})]. \end{aligned} \tag{7.53}$$

The evaluation of the traces yields the results

$$\begin{aligned} [\chi_3]_{\mathrm{n}} &= 0 \\ [\chi_\perp]_{\mathrm{n}} &= \tfrac{1}{4}\sum_{\boldsymbol{k}} \frac{(u_+v_- + u_-v_+)^2}{(\omega_{\boldsymbol{k}^+} + \omega_{\boldsymbol{k}^-})} \end{aligned} \tag{7.54}$$

for the spin susceptibilities, and

$$\begin{aligned} \Gamma^{l}_{33}(\boldsymbol{k}^+, \boldsymbol{k}^-) &= \tfrac{1}{4}[\nabla_{\boldsymbol{k}} h_{\boldsymbol{k}}(u_+v_- + u_-v_+) - \nabla_{\boldsymbol{k}}\Delta_{\boldsymbol{k}}(u_+u_- + v_+v_-)]^2 \\ \Gamma^{l}_{\perp}(\boldsymbol{k}^+, \boldsymbol{k}^-) &= \tfrac{1}{4}[\nabla_{\boldsymbol{k}} h^{+}_{\boldsymbol{k}}(u_+v_- - u_-v_+) - \nabla_{\boldsymbol{k}}\Delta^{-}_{\boldsymbol{k}}(u_+u_- - v_+v_-)]^2 \end{aligned} \tag{7.55}$$

for the spatial correlation functions, where $u_\pm = u_{\boldsymbol{k}\pm}$ and $v_\pm = v_{\boldsymbol{k}\pm}$. Off-diagonal components of the susceptibility and as expected, the susceptibility component of the normal fluid along the twist axis vanishes.

Since the fluctuations in the condensate spin and current are entirely transverse to the magnetisation and the condensate spin current vanishes at long wavelengths, the condensate contribution to long-wavelength spatial spin current fluctuations vanishes, so $[\tilde{\gamma}^{l}]_{\mathrm{n}} = \tilde{\gamma}^{l}$. Let us consider the total normal contribution to the magnetic stiffness $N^{l} - \tilde{\gamma}^{l}$. Since the normal fluid is axially symmetric about the $\hat{z}$ axis, we expect the stiffness of the normal fluid about this axis to vanish

$$[\gamma^{l}_{3}]_{\mathrm{n}} = N^{l}_{3} - \tilde{\gamma}^{l}_{3} = 0. \tag{7.56}$$

This provides a consistency check on the calculation. Integrating expression (7.41) for N^l_{zz} by parts we can rewrite it in the form

$$\begin{aligned} N^l_{zz} &= \frac{-1}{2}\sum_{\boldsymbol{q}}\left[\nabla_l\Delta_{\boldsymbol{q}}\nabla_l\eta_{\boldsymbol{q}} - \nabla_l h_{\boldsymbol{q}}\nabla_l\alpha_{\boldsymbol{q}}\right] \\ &= \sum_{\boldsymbol{q}}\frac{[\nabla_l h_{\boldsymbol{q}}\Delta_{\boldsymbol{q}} - \nabla_{\boldsymbol{q}}\Delta_{\boldsymbol{q}}h_{\boldsymbol{q}}]^2}{4\omega^3_{\boldsymbol{q}}} \\ &= \sum_{\boldsymbol{q}}\frac{[\nabla_l h_{\boldsymbol{q}}2u_{\boldsymbol{q}}v_{\boldsymbol{q}} - \nabla_{\boldsymbol{q}}\Delta_{\boldsymbol{q}}(u^2_{\boldsymbol{q}}+v^2_{\boldsymbol{q}})]^2}{4\omega_{\boldsymbol{q}}} \\ &= \tilde{\gamma}^l(0) \end{aligned} \tag{7.57}$$

confirming that the axial symmetry of the normal fluid is conserved in our calculation. This cancellation does not occur in the normal fluid stiffness perpendicular to the twist axis unless the magnet is bipartite. Indeed, the quantum exchange gap in the single magnon spectrum around $\boldsymbol{q} = \boldsymbol{Q}$, has the effect of *suppressing* the fluctuations perpendicular to the twist axis, reducing $\tilde{\gamma}^l_\perp$ relative to $N^l_\perp$, thereby generating a finite stiffness

$$[\gamma^l_\perp]_{\mathrm{n}} = N^l_\perp - \tilde{\gamma}^l_\perp(\boldsymbol{Q}) > 0. \tag{7.58}$$

Let us finally consider the condensate contribution to the magnetic susceptibility. To do this, in the magnetisation operator we replace

$$\Psi^\dagger{}_{\boldsymbol{q}} \longrightarrow \Psi^\dagger{}_{\boldsymbol{q}} + \langle\Psi^\dagger{}_0\rangle\delta_{\boldsymbol{q}\,0}. \tag{7.59}$$

Since $\langle \boldsymbol{S}(\boldsymbol{x})\rangle = S^*\hat{\boldsymbol{e}}_1(\boldsymbol{x})$, the condensate expectation value is

$$\langle\Psi_0{}^\dagger\rangle = \sqrt{S^*}(1,1,\mathrm{i},-\mathrm{i}). \tag{7.60}$$

Fluctuations in the condensate are transverse to the magnetisation, so the condensate fluctuation magnetisation is

$$\begin{aligned} [\boldsymbol{M}(x)]_{\mathrm{c}} &= M^2(x)_{\mathrm{c}}\,\hat{e}_2(x) + M^3(x)_{\mathrm{c}}\,\hat{e}_3(x) \\ [M^a(\boldsymbol{q})]_{\mathrm{c}} &= \langle\Psi^\dagger{}_0\rangle\tau_3(\sigma^a/2)\Psi_{\boldsymbol{q}} + \Psi^\dagger{}_{-\boldsymbol{q}}\tau_3(\sigma^a/2)\langle\Psi_0\rangle \end{aligned} \tag{7.61}$$

so the condensate spin susceptibility in the twisted coordinates is then

$$[\chi^a(q)]_{\mathrm{c}} = \frac{S^*}{4}\sum_{\boldsymbol{q}\in\frac{1}{2}\mathrm{BZ}}\mathrm{Tr}\big[\sigma^a\Lambda\sigma^a[\mathcal{G}(q)+\mathcal{G}(q)]\big] \qquad (a=2,3) \tag{7.62}$$

where $\Lambda = \langle\Psi_o\rangle\langle\Psi^\dagger\rangle\tau_3 = [1+\sigma^1\otimes\tau_3][1-\sigma^3\otimes\tau_2]$. Evaluating the trace, the condensate susceptibility is found to be

$$[\chi_a]_{\mathrm{c}} = \begin{cases} 0 & (a=1) \\ S^*/(\tilde{h}_{\boldsymbol{Q}} - \Delta_{\boldsymbol{Q}}) & (a=2) \\ S^*/(\tilde{h}_0 + \Delta_0) & (a=3). \end{cases} \tag{7.63}$$

In the large-S limit, the condensate contributions to the stiffness and the susceptibility dominate, and in this limit

$$\chi_a = \begin{cases} 1/[(J(2\boldsymbol{Q}) + J(0))/2 - J(\boldsymbol{Q})] & (a=2) \\ 1/(J(0) - J(\boldsymbol{Q})) & (a=3). \end{cases} \tag{7.64}$$

Taking ratios of the spinwave stiffness (7.28) to the susceptibility, we find

$$[c_a^l]^2 = \gamma_a^l/\chi_a = \begin{cases} S^*[(J(2\boldsymbol{Q}) + J(0))/2 - J(\boldsymbol{Q})]\nabla_l^2 J(\boldsymbol{Q})/2 & (a=2) \\ S^*[(J(0) - J(\boldsymbol{Q}))]\nabla_l^2 J(\boldsymbol{Q})/2 & (a=3). \end{cases} \tag{7.65}$$

These velocities correspond precisely to those found by expanding the spin wave spectrum about $\boldsymbol{q} = 0$ and $\boldsymbol{q} = \boldsymbol{Q}$ respectively.

Finally, let us summarise the combined classical and quantum contributions to the magnetic stiffness and the magnetic susceptibilities in the biaxial helimagnet The stiffness components take the form

$$\gamma_a^l = \begin{cases} N_\perp^l - \tilde{\gamma}_\perp^l & (a=1) \\ [S^*]^2\nabla_l^2 J(\boldsymbol{Q})/2 + N_\perp^l - \tilde{\gamma}_\perp^l & (a=2) \\ [S^*]^2\nabla_l^2 J(\boldsymbol{Q})/2 & (a=3) \end{cases} \tag{7.66}$$

whilst the magnetic susceptibilities are

$$\chi_a = \begin{cases} [\chi_\perp]_\mathrm{n} & (a=1) \\ [S^*/(\tilde{h}_{\boldsymbol{Q}} - \Delta_{\boldsymbol{Q}})] + [\chi_\perp]_\mathrm{n} & (a=2) \\ S^*/(\tilde{h}_0 + \Delta_0) & (a=3). \end{cases} \tag{7.67}$$

Our results show that fluctuations themselves can drive anisotropy and contribute positively to the stiffness of the order parameters. The fluctuations perpendicular to the twist are suppressed by the effects of order from disorder, giving rise to an additional fluctuation contribution to the stiffness about axes perpendicular to the twist. For a bipartite lattice, this quantity is zero. We now go on to discuss the effects of this stiffness in situations where the sublattice magnetisation vanishes.

8. Spin nematic

In the special limit where the spin S is sufficiently small so that the magnetisation drops to zero, the stiffness about the twist axis γ_3^l, vanishes. However, the stiffness about the axes perpendicular to that of the twist remain finite, and are given by (7.58). In this state, the twist plane remains rigid, and rotational invariance is broken via long range order in the four point twist correlation function

$$\langle \boldsymbol{\mathcal{T}}_Y(X) \cdot \boldsymbol{\mathcal{T}}_Y(0)\rangle \longrightarrow |F(Y)\sin(\boldsymbol{Q}\cdot\boldsymbol{Y})|^2 \tag{8.1}$$

where $\boldsymbol{\mathcal{T}}_Y(X) = \boldsymbol{S}(X + Y/2) \times \boldsymbol{S}(X - Y/2)$ is the twist operator. However, the spins in the twist plane have a finite spin correlation length ξ

$$\langle \boldsymbol{S}(Y)\cdot\boldsymbol{S}(0)\rangle = F(Y)(\cos\boldsymbol{Q}\cdot\boldsymbol{Y}) \sim S^2(\cos\boldsymbol{Q}\cdot\boldsymbol{Y})\exp(-|Y|/\xi). \tag{8.2}$$

This state is then a 'spin nematic'. It may be visualised as a helimagnet where quantum fluctuations in the pitch dephase the spins, giving rise to a distribution of magnetic wavevectors with variance $\langle \delta Q^2 \rangle = \xi^{-2}$ (see figure 4.) Unlike a helimagnet, the absence of a magnetisation implies that this state is *translationally invariant.* The residual twist degrees of freedom are now described by an O(3) sigma model

$$I = \frac{\chi}{2} \int \mathrm{d}^d x \, \mathrm{d}t \left\{ (c_l)^2 (\nabla_l \hat{T})^2 - (\partial_t \hat{T})^2 \right\} \tag{8.3}$$

where $\hat{T}(x) = \hat{e}_3(x)$ and $\chi = \chi_a$, $(c_l)^2 = \gamma_a^l / \chi_a$ $(a = 1, 2)$ relate the stiffnesses to those calculated in the last section. A completely analogous sequence of phase transitions is well known in the theory of biaxial nematic liquid crystals. In the phase diagram of nematic liquid crystals, the biaxial phase is separated from the isotropic phase by intermediate uniaxial phases [35]. The presence of a long-range pseudo-scalar order parameter in spin systems has been previously considered by Andreev and Grishchuk [36]. In this particular realisation, we explicitly identify this uniaxial order parameter with a twist associated with incommensurate correlations.

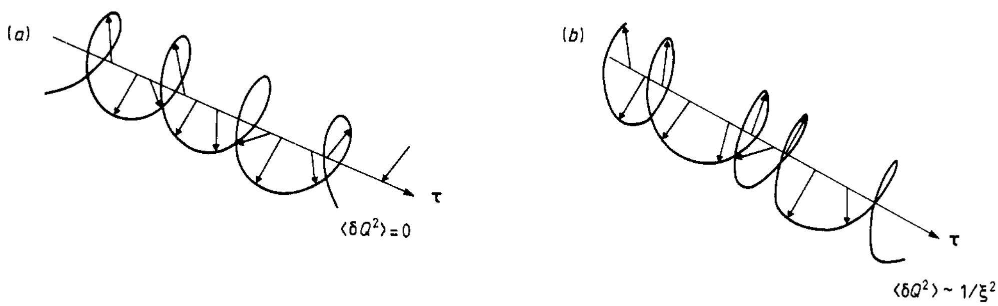

Figure 4. Schematic drawings of (*a*) a helimagnet and (*b*) a spin nematic, these having zero and finite pitch variance, respectively.

In the quantum fluids analogy, this means that the spin boson density S is too small to sustain a spin-Bose condensate, but the paired normal fluid still preserves a twist. Microscopically, the presence of a twist without a magnetisation can be understood from the relationship between the twist and the pairing correlations in the spin fluid. The twist operator may be written in the untwisted reference frame as a product of singlet and triplet pairing fields

$$\boldsymbol{S}_i \times \boldsymbol{S}_j = \tfrac{1}{4}[\boldsymbol{B}^{\dagger}{}_{ij} B_{ij} + B^{\dagger}{}_{ij} \boldsymbol{B}_{ij}] = -\tfrac{1}{4}[\boldsymbol{D}^{\dagger}{}_{ij} D_{ij} + D^{\dagger}{}_{ij} \boldsymbol{D}_{ij}] \tag{8.4}$$

where $(D^{\dagger}{}_{ij}, \boldsymbol{D}^{\dagger}{}_{ij}) = -ib^{\dagger}{}_i(i, \boldsymbol{\sigma})b_j$. In mean field theory this becomes $\langle \boldsymbol{S}_i \times \boldsymbol{S}_j \rangle = \hat{\boldsymbol{n}} \sin(\boldsymbol{Q}.\boldsymbol{R}_{ij}) F(\boldsymbol{R}_{ij})$, where

$$F(\boldsymbol{R}_{ij}) = \frac{1}{2}\chi^{\perp}(\boldsymbol{R}) = \sum_{\boldsymbol{k},\boldsymbol{q}} \frac{1}{2}[\eta_{\boldsymbol{k}-\boldsymbol{q}}\eta_{\boldsymbol{k}} + \alpha_{\boldsymbol{k}-\boldsymbol{q}}\alpha_{\boldsymbol{k}}] \cos \boldsymbol{q} \cdot \boldsymbol{R}_{ij}. \tag{8.5}$$

Thus the presence of a twist and its corresponding stiffness is not linked to the development of a sublattice magnetisation.

Microscopically, the wavefunction for this state is a twisted RVB wavefunction

$$|\Psi\rangle = P_S \exp\left[\sum_{ij} f(\boldsymbol{R}_{ij})[\sin(\boldsymbol{Q}\cdot\boldsymbol{R}_{ij}/2)B^{\dagger}{}_{ij} + \cos(\boldsymbol{Q}\cdot\boldsymbol{R}_{ij}/2)\hat{\boldsymbol{k}}\cdot\boldsymbol{B}^{\dagger}{}_{ij}\right]|0\rangle \qquad (8.6)$$

where $f(\boldsymbol{R}) = \sum_{\boldsymbol{q}} f_{\boldsymbol{q}} \exp(i\boldsymbol{q}\cdot\boldsymbol{R})$ is short-range with $f_o < 1$, and hence there is no sublattice magnetisation. This state is accordingly a 'twisted spin liquid', which breaks parity, but unlike a magnet, it *does not* violate time reversal symmetry. Nevertheless, since the state still breaks rotation symmetry, the gapless twist longitudinal Goldstone mode with velocity c_l will persist even though the sublattice magnetisation is zero.

This spin nematic state is a candidate wavefunction for the spin-$\frac{1}{2}$ frustrated Heisenberg model in the 'disordered' regime where there is no sublattice magnetisation. In principal, such a state can exist even in the absence of a bare third neighbour coupling, due to the renormalisation of $J(Q)$ by fluctuations. Another possibility, motivated by studies of SU(N) magnets, is a dimer ground state [3, 4, 37]. Because our methods are not sensitive to specific spin values, we cannot incorporate both candidates in our treatment at the present time. However, if the mechanism for dimer formation derives from tunneling between different topological configurations of the magnetic moment, a coexistence of dimers with a spin nematic phase is unlikely. The presence of an underlying incommensurate structure will lead to *destructive* interference of the hedgehog configurations that generate the dimerisation, and dimer order will melt on long length scales.

Recent finite size numerical studies of the J_1–J_2–J_3 model indicate that both the twist and the dimer order parameters are large in the intermediate region with no sublattice magnetisation; furthermore $\chi_{S\times S}$ scales properly with the system size [38] and is further enhanced when J_3 is increased from zero. A variety of groups have also found some evidence for static valence bond ground states in the same regime [38, 39]. A possible explanation of these seemingly contradictory results it that the short-range dimer order persists up to length scales comparable with the quantum exchange length. More extensive numerical work is needed to decide these issues.

9. Discussion

In conclusion, we have presented a quantum fluids approach to incommensurate magnetism. Exploiting the *local* gauge symmetry associated with the conservation of spin, we have treated the quantum Heisenberg antiferromagnet as a spin superfluid. This analogy operates on both short and long length scales; for example, there are spin analogues for both rotons and the Josephson equation. Table 1 in section 1 summarises the most important parallels between antiferromagnets and neutral superfluids. The aim of the present work has been to develop a rotationally invariant approach to the general class of helimagnet; such a treatment must capture the essential physics of spin fluctuations, in particular the generalised self-consistent Weiss exchange field and Villain's fluctuation-stabilised order. Here we check the results of our extended Schwinger boson technique whenever possible with those of known methods, specifically spin wave theory (to order $1/S^2$) and Polyakov scaling, before studying cases where global spin rotation invariance is partially or fully restored. In particular, our pairing equations are exact in the large-S limit, and our results reproduce the classical Goldstone mode structure, the quantum exchange gaps and the Ising transition temperature [30] derived from scaling.

Within a spin superfluid picture of Heisenberg antiferromagnetism, the spin fluctuations are described by the normal fluid and classical magnetism is the condensate. Following Villain, we emphasise that spin fluctuations can select new forms of long-range order *independent* of moment development; the fluctuation-selection of a twisted state from the manifold of classically degenerate O(6) magnets provides an example. Specifically, we show that in a helimagnet the two fluids *each* have *uniaxial* order, one associated with the twist and the other related to the sublattice magnetisation. The fluctuation-stabilised helimagnet then has *biaxial* order; it can melt to an isotropic state via *two* uniaxial phases, the Neel magnet or the spin nematic, [15], completely analogous to the biaxial–uniaxial transition of liquid crystals [40]. The spin nematic has the interesting property of having a broken O(3) symmetry *without* breaking time-reversal symmetry; specifically *non-local* spin order parameters exist in the absence of a local moment.

There are several experimental signatures of a spin nematic. At low temperatures, the twist Goldstone modes will lead to a power-law specific heat capacity $C_V \sim T^d$. In two dimensions, the development of long-range twist correlations will be accompanied by a peak in the specific heat capacity at temperatures of order the quantum exchange energy. The elastic neutron scattering will contain no Bragg peak, but will be characterised by a broad Lorentzian maximum around the incommensurate magnetic wavevector $\pm \boldsymbol{Q}_o$

$$S(\boldsymbol{q}) \sim (1/\pi\xi)[(\boldsymbol{q} \pm \boldsymbol{Q}_o)^2 + \xi^{-2}]. \tag{9.1}$$

The usual Born scattering of neutrons is only sensitive to the *two*-point spin correlations; in principle, however, small-angle multiple neutron scattering can be used to probe the *four*-spin twist–twist correlation function. A magnetic field applied parallel to the twist axis of the nematic will induce a net chirality $\boldsymbol{S}_i \cdot (\boldsymbol{S}_j \times \boldsymbol{S}_j)$, which is known to generate a left-right anisotropy in the scattering of polarised neutrons ($\sigma_{sc} \sim (\boldsymbol{k} \times \boldsymbol{k}') \cdot \boldsymbol{\mu}_{\mathrm{n}}$) [41].

P-type spin nematics, like cholesterics and helimagnets, will be optically active. The magnetic order in a spin nematic decays on a slow time scale $t \sim \xi/c_{\mathrm{s}}$, where c_{s} is a spin wave velocity. Therefore, given single spin nematic domains of size L, where $L \ll \xi(c/c_{\mathrm{s}}) \sim 10^4\xi$, electromagnetic radiation will perceive a spin nematic as a disordered helimagnet with distribution of pitch lengths. Following (7.19), the magnetic permittivity tensor will contain a non-uniform component of the form

$$\delta\mu^{\alpha\beta}(\boldsymbol{R}) \sim \mu_o(\chi_1 - \chi_2) e_1^\alpha(\boldsymbol{R}) e_1^\beta(\boldsymbol{R}) \tag{9.2}$$

and, as in cholesterics, this generates optical activity. It is then straightforward to extend the standard liquid crystal analysis. The basic propagation equation is

$$\nabla^2 \boldsymbol{E}(x) = -(\omega^2/c^2)\mu(x) \cdot \boldsymbol{E}(x). \tag{9.3}$$

Following de Vries [42], for an incident plane-polarised electromagnetic wave parallel or antiparallel to the twist axis, the optical rotation per length L is

$$\phi(k) \sim \int \mathrm{d}Q\, S(Q) \frac{QL}{32} \left[\frac{\mu_o(\chi_1 - \chi_2)}{\mu}\right]^2 \frac{k^4}{Q^2(Q^2 - k^2)} \tag{9.4}$$

where $S(\boldsymbol{q}) \sim 1/\pi\xi[(\boldsymbol{q} - \boldsymbol{Q}_o)^2 + \xi^{-2}]$. Like the cholesterics case, the rotation handedness is *independent* of the direction of propagation, as opposed to a Faraday rotation.

Furthermore, a plane-polarised beam reflected from a spin nematic will acquire a circularly polarised component with the same handedness as the nematic [42]. For optical frequencies the dichroism is weak and scales as $\sim\omega^4$, but at x-ray frequencies, as in helimagnets, we expect the phenomenon to become strongly resonant [43, 44].

We are aware of two distinct possible realisations of spin nematic behaviour in nature. The first is provided by the nuclear magnetism of two dimensional He^3 films adsorbed on graphite. Recently, Elser [45] has postulated that these nuclear spins can be described by a $S = \frac{1}{2}$ Heisenberg model on a 'Kagome' lattice, for which the classical ground state is infinitely degenerate. Another example is the recently discovered chromium ($S = \frac{3}{2}$) magnetoplumbite compound $SrCr_{8-x}Ga_{4+x}O_{19}$; it consists of parallel planes of chromium atoms arranged on a Kagome lattice [46]. This system does not order magnetically, and displays a T^2 specific heat [47, 48], as expected for a two dimensional spin nematic. A close comparison of theory and experiment is in progress. [48, 49]. Though our calculations have been specific to two dimensions, they should apply equally well to known three-dimensional helimagnets, where there is the possibility of finite temperature spin nematic phases above the Curie temperature. Frequently, these systems make the transition to a commensurate magnet by an unraveling of the spiral order. Those cases where the unraveling does not occur, or is only partial, such as erbium, holmium or dysprosium are good candidates for a finite temperature spin nematic.

Our discussion has been limited to the case of pure magnets; however some of the techniques developed here may be useful in the study of doped Mott antiferromagnetic insulators. There, charge fluctuations drive an incommensuration in the magnetic structure [50]; though several attempts have been made to describe these twisted phases with Schwinger bosons, the use of mixed parity pairing, a necessity in order to recover the correct large S Goldstone mode structure, has not yet been employed.

We end with the amusing possibility of a superconducting spin nematic; a spin nematic violates inversion symmetry but not time-reversal symmetry, and thus can can readily coexist with BCS pairing, forming an incommensurate superconductor. Such a superconductor would involve *mixed* parity pairing due to the presence of the magnetic twist [51]. Since this state violates PT symmetry, the Landau–Ginzberg theory permits terms which directly couple the vector potential to the magnetic field. The presence of a magnetic field perpendicular to the twist axis should result in the flow of charge along the direction of the $\boldsymbol{Q}$ vector. We hope to investigate these possibilities in future work.

Note added in proof. After submission of this paper we became aware of related work by Schulz [52] and Nersesyan and Luther [53] on spin nematic behaviour in generalised Hubbard models [54].

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Appendix

Here we derive expressions for the paramagnetic spin currents. We begin with the field dependent spin boson Hamiltonian in the untwisted reference frame

$$\hat{H}'[A] = \sum_{\boldsymbol{q}\in\frac{1}{2}\mathrm{BZ}} \tilde{\Psi}^{\dagger}{}_{\boldsymbol{q}}\tau_3\mathcal{H}[\boldsymbol{q}_l + (Q_l\sigma^3 - \boldsymbol{A}_l\cdot\boldsymbol{\sigma})/2]\tilde{\Psi}_{\boldsymbol{q}} - \sum_{\boldsymbol{q}} \boldsymbol{B}_{\boldsymbol{q}}\cdot\boldsymbol{S}_{-\boldsymbol{q}}. \qquad \text{(A1)}$$

Expanding this expression in a gradient expansion in A, then to first order in A

$$\hat{H}'[A] = \hat{H}' - \sum_{\boldsymbol{R}} \boldsymbol{A}_\mu(R)\cdot\boldsymbol{j}'_\mu(R) \qquad \text{(A2)}$$

where $A_\mu = (\boldsymbol{B}, \boldsymbol{A}_l)$, $\boldsymbol{j}_\mu = (\boldsymbol{M}, \boldsymbol{j}_l)$ and the paramagnetic spin current is $\boldsymbol{j}_l(R) = \sum_{\boldsymbol{q}} \boldsymbol{j}_l(\boldsymbol{q})\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{R})$, where

$$\boldsymbol{j}_l(\boldsymbol{q}) = \sum_{\boldsymbol{k}\in\frac{1}{2}\mathrm{BZ}} \tilde{\Psi}^{\dagger}{}_{\boldsymbol{k}-\boldsymbol{q}/2}\tau_3\left\{\frac{\boldsymbol{\sigma}}{4}, \nabla_{\boldsymbol{q}}\mathcal{H}[\boldsymbol{q}]\right\}\tilde{\Psi}_{\boldsymbol{q}+\boldsymbol{k}/2}. \qquad \text{(A3)}$$

Here the curly parentheses denote an anticommutator. Resolving the spin currents along the *fixed* axes $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}})$, $\boldsymbol{j}_l(R) = j_l'^1\hat{\boldsymbol{\imath}} + j_l'^2\hat{\boldsymbol{\jmath}} + j_l'^3\hat{\boldsymbol{k}}$, then the components $j_l'^a(\boldsymbol{q})$ are given by

$$\begin{aligned} j_l'^3(\boldsymbol{q}) &= \sum_{\boldsymbol{k}\in\frac{1}{2}\mathrm{BZ}} \tilde{\Psi}^{\dagger}{}_{\boldsymbol{k}-\boldsymbol{q}/2}\tau_3\frac{\sigma^3}{2}\nabla_{\boldsymbol{k}}\mathcal{H}(\boldsymbol{k} - \sigma^3\boldsymbol{Q}/2)\tilde{\Psi}_{\boldsymbol{k}+\boldsymbol{q}/2} \\ j_l'^{\pm}(\boldsymbol{q}) = j_l'^1(\boldsymbol{q}) \pm \mathrm{i}j_l'^2(\boldsymbol{q}) &= \sum_{\boldsymbol{k}\in\frac{1}{2}\mathrm{BZ}} \tilde{\Psi}^{\dagger}{}_{\boldsymbol{k}-\boldsymbol{q}/2}\tau_3\sigma^{\pm}\nabla_{\boldsymbol{k}}\tilde{\mathcal{H}}(\boldsymbol{k} - \sigma^3\boldsymbol{Q}/2)\tilde{\Psi}_{\boldsymbol{k}+\boldsymbol{q}/2} \end{aligned} \qquad \text{(A4)}$$

where $\sigma^{\pm} = (\sigma^1 \pm \mathrm{i}\sigma^2)/2$ and

$$\tilde{\mathcal{H}}(\boldsymbol{q}) = [h^+(\boldsymbol{q})\ \tau_3 + \mathrm{i}\Delta^-(\boldsymbol{q})\ \tau_1] \qquad \text{(A5)}$$

where

$$\begin{aligned} h_{\boldsymbol{q}}^+ &= \tfrac{1}{2}(h_{\boldsymbol{q}+\boldsymbol{Q}/2} + h_{\boldsymbol{q}-\boldsymbol{Q}/2}) \\ \Delta_{\boldsymbol{q}}^- &= \tfrac{1}{2}(\Delta_{\boldsymbol{q}+\boldsymbol{Q}/2} - \Delta_{\boldsymbol{q}-\boldsymbol{Q}/2}). \end{aligned} \qquad \text{(A6)}$$

Let us now resolve the spin curents along the precessing axes $\hat{e}_\lambda(R)$

$$\boldsymbol{j}_\mu(\boldsymbol{R}) = \sum_a j_\mu^a(\boldsymbol{R})\hat{e}_a(\boldsymbol{R}) \qquad \text{(A7)}$$

where

$$j_\mu^\lambda(\boldsymbol{R}) = \boldsymbol{j}_\mu(\boldsymbol{R})\hat{e}_\lambda(\boldsymbol{R}). \tag{A8}$$

Taking the Fourier transform of these components,

$$\begin{aligned} j_\mu^3(\boldsymbol{q}) &= j_\mu^{'3}(\boldsymbol{q}) \\ j_\mu^\pm(\boldsymbol{q}) &= \sum_{\boldsymbol{R}} \boldsymbol{j}_\mu(\boldsymbol{R}) \cdot [\hat{e}_1(\boldsymbol{R}) \pm \mathrm{i}\hat{e}_2(\boldsymbol{R})] \exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{R}) \\ &= \sum_{\boldsymbol{R}} j_\mu^{'\pm}(\boldsymbol{R}) \exp[-\mathrm{i}(\boldsymbol{q} \pm \boldsymbol{Q})\cdot\boldsymbol{R}] \\ &= j^{'\pm}(\boldsymbol{q} \pm \boldsymbol{Q}). \end{aligned} \tag{A9}$$

Finally, noting that

$$\tilde{\Psi}^\dagger{}_{\boldsymbol{k}} = \sum_\sigma \Psi^\dagger{}_{\boldsymbol{k}+\sigma\boldsymbol{Q}/2}\mathcal{P}_\sigma \qquad (\sigma = \pm 1) \tag{A10}$$

where $\mathcal{P}_\sigma = [1 + \sigma\sigma^3]/2$ projects out the up and down spin components of the spinor, we note that for the spin independent matrix $\tau_3\tilde{\mathcal{H}}_{\boldsymbol{k}}$

$$\tilde{\Psi}^\dagger{}_{\boldsymbol{k}-(\boldsymbol{q}\pm\boldsymbol{Q})/2}\sigma^\pm\tau_3\tilde{\mathcal{H}}_{\boldsymbol{k}}\tilde{\Psi}_{\boldsymbol{k}+(\boldsymbol{q}\pm\boldsymbol{Q})/2} = \Psi^\dagger{}_{\boldsymbol{k}-\boldsymbol{q}/2}\sigma^\pm\tau_3\tilde{\mathcal{H}}_{\boldsymbol{k}}\Psi_{\boldsymbol{k}+\boldsymbol{q}/2} \tag{A11}$$

whilst

$$\sum_{\boldsymbol{k}\in\frac{1}{2}\mathrm{BZ}} \tilde{\Psi}^\dagger{}_{\boldsymbol{k}-\boldsymbol{q}/2}\sigma^3\mathcal{H}(\boldsymbol{k} + \sigma^3\boldsymbol{Q}/2)\tilde{\Psi}_{\boldsymbol{k}+\boldsymbol{q}/2} = \sum_{\boldsymbol{k}\in\frac{1}{2}\mathrm{BZ}} \Psi^\dagger{}_{\boldsymbol{k}-\boldsymbol{q}/2}\sigma^3\mathcal{H}(\boldsymbol{k})\tilde{\Psi}_{\boldsymbol{k}+\boldsymbol{q}/2}. \tag{A12}$$

Using these results, the spin currents can be re-written in terms of the Bose fields in the twisted frame

$$\begin{aligned} j_l^3(\boldsymbol{q}) &= \sum_{\boldsymbol{k}\in\frac{1}{2}\mathrm{BZ}} \Psi^\dagger{}_{\boldsymbol{k}-\boldsymbol{q}/2}\tau_3\frac{\sigma^3}{2}\nabla_k\mathcal{H}(\boldsymbol{k})\Psi_{\boldsymbol{k}+\boldsymbol{q}/2} \\ j_l^\pm(\boldsymbol{q}) = j_l^1(\boldsymbol{q}) \pm ij_l^2(\boldsymbol{q}) &= \sum_{\boldsymbol{k}\in\frac{1}{2}\mathrm{BZ}} \Psi^\dagger{}_{\boldsymbol{k}-\boldsymbol{q}/2}\tau_3\sigma^\pm\nabla_k\tilde{\mathcal{H}}(\boldsymbol{k})\Psi_{\boldsymbol{k}+\boldsymbol{q}/2}. \end{aligned} \tag{A13}$$

Written out more explicitly in terms of the individual Bose fields $b^\dagger{}_{\boldsymbol{q}\sigma}$, the spin currents are

$$j_l^3(\boldsymbol{q}) = \frac{1}{2}\sum_{\boldsymbol{k}\sigma}\nabla_k h_{\boldsymbol{k}}(b^\dagger{}_{\boldsymbol{k}-\,\sigma}\sigma b_{\boldsymbol{k}+\,\sigma}) - \frac{1}{2}\sum_{\boldsymbol{k}}\nabla_{\boldsymbol{k}}\Delta_{\boldsymbol{k}}(b^\dagger{}_{\boldsymbol{k}-\uparrow}b^\dagger{}_{-\boldsymbol{k}+\,\downarrow} + b_{-\boldsymbol{k}-\downarrow}b_{\boldsymbol{k}+\uparrow})$$

$$j_l^+(\boldsymbol{q}) = \sum_{\boldsymbol{k}}\left[\nabla_k h_{\boldsymbol{k}}^+(b^\dagger{}_{\boldsymbol{k}-\,\uparrow}b_{\boldsymbol{k}+\downarrow}) - \frac{1}{2}\nabla_{\boldsymbol{k}}\Delta_{\boldsymbol{k}}^-(b^\dagger{}_{\boldsymbol{k}-\,\uparrow}b^\dagger{}_{-\boldsymbol{k}+\,\uparrow} - b_{-\boldsymbol{k}-\,\downarrow}b_{\boldsymbol{k}+\,\downarrow})\right] \tag{A14}$$

$$j_l^-(\boldsymbol{q}) = \sum_{\boldsymbol{k}}\left[\nabla_k h_{\boldsymbol{k}}^+(b^\dagger{}_{\boldsymbol{k}-\downarrow}b_{\boldsymbol{k}+\uparrow}) - \frac{1}{2}\nabla_{\boldsymbol{k}}\Delta_{\boldsymbol{k}}^-(b^\dagger{}_{\boldsymbol{k}-\,\downarrow}b^\dagger{}_{-\boldsymbol{k}+\,\downarrow} - b_{-\boldsymbol{k}-\,\uparrow}b_{\boldsymbol{k}+\uparrow})\right] \tag{A14}$$

where $\boldsymbol{k}^{\pm} = \boldsymbol{k} \pm \boldsymbol{q}/2$. Similar expressions also hold for $\boldsymbol{j}_0 = \boldsymbol{M}$, when $\nabla_{\boldsymbol{k}}\mathcal{H}$ and $\nabla_{\boldsymbol{k}}\tilde{\mathcal{H}}$ are replaced by the identity.

References

[1] Anderson P W 1987 *Science* **235** 1196
Anderson P W 1988 *Frontiers in Many Particle Physics (International School of Physics)* ed J R Schrieffer and R A Broglia (Amsterdam: North-Holland)
[2] Affleck I, Kennedy T, Leib E and Tanaka H 1987 *Phys. Rev. Lett.* **59** 799
[3] Read N and Sachdev S 1989 *Phys. Rev. Lett.* **62** 1694
[4] Haldane F D M 1988 *Phys. Rev. Lett.* **61** 1029
[5] Liang S, Doucot B and Anderson P W 1988 *Phys. Rev. Lett.* **61** 365
[6] Arovas D P and Auerbach A 1988 *Phys. Rev.* B **38** 316
[7] Yoshioka D 1989 *J. Phys. Soc. Japan* **58** 3733
[8] Takahashi M 1986 *Prog. Theor. Phys. Suppl.* **87** 233
[9] Schwinger J 1952 On angular momentum *US Atomic Energy Commission* NYO-3071 (unpublished)
[10] Sarker S, Jayaprakash C, Krishnamurthy H R and Ma M 1989 *Phys. Rev.* B **40** 5028
[11] Gulayev Y V 1965 *JETP Lett.* **2** 1
[12] Halperin B I and Hohenberg P C 1969 *Phys. Rev.* **188** 898
[13] Enz C P 1974 *Rev. Mod. Phys.* **46** 704
[14] Anderson P W and Brinkman W F 1975 *Proc. 15th Scottish Summer School in Physics* ed J Armytage and I Farquhar (New York: Academic) p 317
[15] Andreev A F and Marchenko V I 1980 *Sov. Phys.-Usp.* **23** 21
[16] Haldane F D M 1983 *Phys. Lett.* **93A** 464
[17] Note that $S = \frac{1}{2}$ Bose fluids are feasible in non-relativistic systems: a spin polarised hydrogen is an example of such. The long-wavelength magnetic modes of this triplet paired fluid are described by an SO(3) sigma model. See:
Siggia E D and Ruckenstein A E 1980 *Phys. Rev. Lett.* **44** 1474
[18] Brout R and Thomas H 1967 *Physics* **3** 317
[19] Scalapino D and Hone D 1989 private communication
[20] Onsager L 1936 *J. Am. Chem. Soc.* **58** 1486
[21] Takahashi M 1989 *Phys. Rev.* B **40** 2494
[22] Shendar E 1982 *Sov. Phys.-JETP* **56** 178
Gukasov A G, Bruckel Th, Dorner B, Plakhty V D, Prandl W, Shender E F and Smirnov O P 1988 *Europhys. Lett.* **7** 83
[23] Ioffe L B and Larkin A I 1988 *Mod. Phys.* B **2** 203
[24] Cooper B R, Elliott R J, Nettel S J and Suhl H 1962 *Phys. Rev.* **127** 57
[25] Villain J 1977 *J. Physique* **38** 26
Villain J, Bidaux R, Carton J P and Conte R 1980 *J. Physique* **41** 1263
[26] Henley C L 1989 *Phys. Rev. Lett.* **62** 2056
Long M W 1989 *J. Phys.: Condens. Matter* **1** 2857
[27] The existence of twisted magnetic structures was predicted simultaneously by Villain, Kaplan and Yoshimori in the following three references.
Villain J 1959 *J. Phys. Chem. Solids* **11** 303
Kaplan T A 1959 *Phys. Rev.* **116** 88
Yoshimori A *J. Phys. Soc. Japan* **14** 807
[28] For a detailed study of twisted helicoidal magnetic structures see:
Dzyaloshinski I E 1964 *Sov. Phys.-JETP* **19** 1960; 1965 *Sov. Phys.-JETP* **20** 223; 665
[29] Chandra P and Doucot B *Phys. Rev.* B **38** 9335
[30] Chandra P, Coleman P and Larkin A I 1990 *Phys. Rev. Lett.* **64** 88
[31] Chakravarty S, Halperin B I and Nelson D R 1988 *Phys. Rev. Lett.* **60** 1057
[32] Pimpanelli A, Rastelli E and Tassi A 1989 *J. Phys.: Codens. Matter* **1** 2131
[33] Chubukov A I 1984 *J. Phys. C: Solid State Phys.* **17** L991
[34] Note that if one takes a long-wavelength average of the susceptibilities, in the magnetisation plane, setting $\gamma^{\mu}_{-} = 0$, the large non-uniform response is effectively removed from the linear response theory. See:

Dombre T and Read N 1989 *Phys. Rev.* B **39** 6797
[35] Weigman P B, Larkin A I and Filev V I 1975 *Sov. Phys.-JETP* **41** 944
[36] Andreev A F and Grishchuk I A 1984*Sov. Phys.-JETP* **60** 267
[37] Bhatt R and Sachdev S 1990 *Preprint* AT&T Bell Laboratories
[38] Dagotto E and Moreo A 1989 *Phys. Rev. Lett.* **63** 2148
[39] Gelfand M, Singh R and Huse D A 1989 *Preprint* AT&T Bell Laboratories
[40] Weigman P B, Larkin A I and Filev V M 1975 *Sov. Phys.-JETP* **41** 944
[41] Maléev S V 1984 *Sov. Phys.-JETP* **59** 366
Lazuta A V, Maléev S V, and Toperverg B P 1978 *Phys. Lett.* **65A** 438; 1978 *Sov. Phys.-JETP* **48** 386; 1981 *Sov. Phys.-JETP* **54** 1113
[42] de Vries H 1951 *Acta Crystallogr.* **4** 219
de Gennes P G 1974 1974 *Physics of Liquid Crystals* (Oxford: Oxford University Press) pp 222–35
[43] Blume M and Gibbs D 1974 *Phys. Rev.* B **47** 1779
Blume M 1985 *J. Appl. Phys.* **57** 3615
[44] Siddons D P, Hart M, Amemiya Y and Hastings J B 1990 *Phys. Rev. Lett.* **64** 1967
[45] Elser V 1989 *Phys. Rev. Lett.* **20** 2405
[46] Obradors X, Labarta A, Isalgué A, Tejada J, Rodriguez J and Pernet M 1988 *Solid State Commun.* **65** 189
[47] Ramirez A P, Espinosa G P and Cooper A S 1990 *Phys. Rev. Lett.* **64** 2070
[48] Broholm C, Aeppli G, Ramirez A P, Epinosa G P and Cooper A S 1990 to be published
[49] Ritchey I and Coleman P 1990 to be published
[50] Shraiman B I and Siggia E D 1989 *Phys Rev Lett.* **62** 1564
[51] Chandra P and Coleman P 1989 Int. J. Mod. Phys. **3** 1729
[52] Schultz H J 1989 *Phys. Rev.* B **39** 2940
[53] Nersesyan A A and Luther A 1989 unpublished
[54] The low-temperature magnetic properties of thin n-type spin nematics are discussed in Nersesyan A A, Japardize G I and Kimeridge I G 1990 *J. Phys. Condens. Matter* submitted